MBA Teaching Note 07-03
Derivations of Present Value Formulas

Finance students learn several key present value formulas. For the most part they can learn what they need to know in introductory classes without knowing where the formulas come from. But some students are curious and others do not believe in a formula until they know where it comes from. This note shows how the major present value formulas are derived.

Present Value of an Annuity

Consider an annuity of N payments of C dollars each. We know that the present value is

$$\begin{aligned} V_0 &= \frac{C}{(1+r)^1} + \frac{C}{(1+r)^2} + \ldots + \frac{C}{(1+r)^N} \\ &= C\left(\frac{1}{(1+r)^1} + \frac{1}{(1+r)^2} + \ldots + \frac{1}{(1+r)^N}\right) \\ &= C\sum_{i=1}^{N}\frac{1}{(1+r)^i}. \end{aligned}$$

We are told that this formula simplifies to

$$V_0 = C\left(\frac{1-(1+r)^{-N}}{r}\right).$$

Now let us see how this formula is obtained. Rewrite the above equation and number it:

$$V_0 = C\left(\frac{1}{(1+r)^1} + \frac{1}{(1+r)^2} + \ldots + \frac{1}{(1+r)^{N-1}} + \frac{1}{(1+r)^N}\right) \tag{1}$$

Divide the left- and right-hand sides 1 + r:

$$\frac{V_0}{1+r} = C\left(\frac{1}{(1+r)^2} + \frac{1}{(1+r)^3} + \ldots + \frac{1}{(1+r)^N} + \frac{1}{(1+r)^{N+1}}\right). \tag{2}$$

Subtract equation (2) from equation (1):

$$V_0 - V_0\left(\frac{1}{1+r}\right) = C\left(\frac{1}{1+r} - \frac{1}{(1+r)^{N+1}}\right).$$

Solve this equation for V_0:

$$V_0\left(1-\frac{1}{1+r}\right) = C\left(\frac{1}{1+r} - \frac{1}{(1+r)^{N+1}}\right)$$

$$V_0\frac{1+r-1}{1+r} = C\left(\frac{1}{1+r} - \frac{1}{(1+r)^{N+1}}\right)$$

$$V_0\left(\frac{r}{1+r}\right) = C\left(\frac{1}{1+r} - \frac{1}{(1+r)^{N+1}}\right)$$

(Multiply by (1+r))

$$V_0 r = C\left(1-\frac{1}{(1+r)^N}\right)$$

$$V_0 = C\left(\frac{1-(1+r)^{-N}}{r}\right).$$

Of course, this formula is the discount factor for an annuity. A similar approach can be taken to derive the compound factor for an annuity.

A perpetuity is a special case of an annuity with an infinite number of payments. Using the annuity discount formula above, we simply let N approach infinity. In that case $(1 + r)^{-\infty}$ will approach zero, leaving,

$$V_0 = \frac{1}{r}.$$

Constant Growth Model

This model is widely used to find the price of a stock. Starting at time 0, we let the next cash flow be specified as C_1. Each succeeding dividend is higher than the preceding one by a factor $(1 + g)$. This growth continues forever. The present value is

$$V_0 = \frac{C_1}{1+r} + \frac{C_1(1+g)}{(1+r)^2} + \frac{C_1(1+g)^2}{(1+r)^3} + \ldots + \frac{C_1(1+g)^{\infty-1}}{(1+r)^{\infty}} + \frac{C_1(1+g)^{\infty}}{(1+r)^{\infty+1}}. \quad (3)$$

Now let us multiply this equation by $(1 + g)/(1 + r)$:

$$V_0\left(\frac{1+g}{1+k}\right) = \frac{C_1(1+g)}{(1+r)^2} + \frac{C_1(1+g)^2}{(1+r)^3} + \frac{C_1(1+g)^3}{(1+r)^4} + \ldots + \frac{C_1(1+g)^{\infty}}{(1+r)^{\infty+1}} + \frac{C_1(1+g)^{\infty+1}}{(1+r)^{\infty+2}}. \quad (4)$$

Now subtract equation (4) from equation (3):

$$V_0 - V_0\left(\frac{1+g}{1+k}\right) = \frac{C_1}{1+r} + \frac{C_1(1+g)}{(1+r)^2} + \frac{C_1(1+g)^2}{(1+r)^3} + \ldots + \frac{C_1(1+g)^{\infty-1}}{(1+r)^{\infty}} + \frac{C_1(1+g)^{\infty}}{(1+r)^{\infty+1}}$$

$$-\left(\frac{C_1(1+g)}{(1+r)^2} + \frac{C_1(1+g)^2}{(1+r)^3} + \frac{C_1(1+g)^3}{(1+r)^4} + \ldots + \frac{C_1(1+g)^{\infty}}{(1+r)^{\infty+1}} + \frac{C_1(1+g)^{\infty+1}}{(1+r)^{\infty+2}}\right)$$

$$V_0\left(1 - \frac{1+g}{1+k}\right) = \frac{C_1}{1+r} - \frac{C_1(1+g)^{\infty+1}}{(1+r)^{\infty+2}}.$$

At this point we make the assumption that the growth rate is less than the discount rate. If that is the case, then the second term on the right-hand side is equal to zero, leaving[1]

$$V_0\left(\frac{1+r-(1+g)}{1+r}\right) = \frac{C_1}{1+r}$$

$$V_0\left(\frac{r-g}{1+r}\right) = \frac{C_1}{1+r}$$

$$V_0 = \frac{C_1}{r-g}.$$

Of course, this is the formula we are interested in. Typically, the cash flow is a dividend (replace C_1 with D_1) and sometimes the discount rate is expressed as k or another symbol. The present value is usually replaced by the price symbol, P_0.

The perpetuity is a case of zero growth. Thus, let $g = 0$ and the above formula simplifies to the well-known perpetuity formula as shown above.

Constant Growth for a Finite Period of Time

In the above formula, the constant growth continues forever. In some problems, constant growth occurs for only a finite period of time, such as the following.

$$V_0 = \frac{C_1}{1+r} + \frac{C_1(1+g)}{(1+r)^2} + \frac{C_1(1+g)^2}{(1+r)^3} + \ldots + \frac{C_1(1+g)^{N-2}}{(1+r)^{N-1}} + \frac{C_1(1+g)^{N-1}}{(1+r)^{N}}. \quad (5)$$

[1]The second term on the right-hand side involves raising a factor $(1 + g)$ to the power ∞ - 1 divided by a larger factor, $(1 + r)$, raised to a higher power, ∞. When the power of the denominator gets large enough, the denominator swamps the numerator, resulting in a value of zero for that term.

Note in this problem that there are N cash flows, where the N^{th} cash flow is higher than the first by the factor $(1 + g)^{N-1}$. Now, multiply equation (5) by $(1 + g)/(1 + r)$:

$$V_0\left(\frac{1+g}{1+r}\right)=\frac{C_1(1+g)}{(1+r)^2}+\frac{C_1(1+g)^2}{(1+r)^3}+\frac{C_1(1+g)^3}{(1+r)^4}+...+\frac{C_1(1+g)^{N-1}}{(1+r)^N}+\frac{C_1(1+g)^N}{(1+r)^{N+1}}. \qquad (6)$$

Subtract equation (6) from equation (5):

$$V_0-V_0\left(\frac{1+g}{1+r}\right)=\frac{C_1}{1+r}+\frac{C_1(1+g)}{(1+r)^2}+\frac{C_1(1+g)^2}{(1+r)^3}+...+\frac{C_1(1+g)^{N-1}}{(1+r)^N}$$
$$-\left(\frac{C_1(1+g)}{(1+r)^2}+\frac{C_1(1+g)^2}{(1+r)^3}+\frac{C_1(1+g)^3}{(1+r)^4}+...+\frac{C_1(1+g)^{N-1}}{(1+r)^N}+\frac{C_1(1+g)^N}{(1+r)^{N+1}}\right)$$
$$V_0\left(1-\frac{1+g}{1+r}\right)=\frac{C_1}{1+r}-\frac{C_1(1+g)^N}{(1+r)^{N+1}}.$$

Now, solve for V_0:

$$V_0\left(\frac{1+r-(1+g)}{1+r}\right)=C_1\left(\frac{1}{1+r}-\frac{(1+g)^N}{(1+r)^{N+1}}\right)$$
$$V_0\left(\frac{r-g}{1+r}\right)=C_1\left(\frac{1}{1+r}-\frac{(1+g)^N}{(1+r)^{N+1}}\right)$$
$$V_0(r-g)=C_1\left(1-\frac{(1+g)^N}{(1+r)^N}\right)$$
$$V_0=C_1\frac{\left(1-\left(\frac{1+g}{1+r}\right)^N\right)}{r-g}.$$

This formula is not specifically covered in my class, but it can be used when there is constant growth for a finite period of time. Note that if N is allowed to be infinite and provided $g < r$, then the expression $((1 + g)/(1 + r))^{\infty}$ goes to zero and the formula reduces to the same one we previously covered for the case of constant growth at $g < r$ forever.