TEACHING NOTE 00-01:
LINEAR HOMOGENEITY, EULER'S RULE, THE BLACK-SCHOLES MODEL, AND AN APPLICATION TO FORWARD START OPTIONS

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A common derivation of the Black-Scholes model is obtained by combining a unit of stock and a certain number of short positions in call options such that the risk of stock price changes is eliminated, leaving a risk-free position with a return equal to the risk-free rate. Given knowledge of all other input variables, the call price is obtained via the solution of a partial differential equation. Details are provided in TN99-02. In this teaching note, we look at an alternative and little-known derivation that makes use of the special property of linear homogeneity of a European option price. First let us review the concept of a homogeneous function.

Homogeneous Functions

A function f(x,y,z) is said to be homogeneous of degree k with respect to the variables x and y if the following condition holds

$$f(\lambda x, \lambda y, z) = \lambda^k f(x, y, z).$$

In other words if variables x and y are each increased by a factor λ and the function value increases by a factor λ^k, then the function is said to be homogeneous of degree k. A function such as the above can be homogeneous with respect to one or more variables. Note that the function specified above is not homogeneous with respect to variable z.

The simplest case of homogeneity is of degree zero. Consider the simple function f(x,y,z) = xz/y. Then note what happens when we multiply x and y by λ:

$$f(\lambda x, \lambda y, z) = \frac{\lambda xz}{\lambda y} = \frac{xz}{y}.$$

This function is unaffected by changing x and y by the factor λ and is, thus, homogeneous of degree zero with respect to x and y because

$$f(\lambda x, \lambda y, z) = \lambda^0 f(x, y, z) = f(x, y, z).$$

The function

$$f(x, y, z) = z\sqrt{xy}$$

is homogeneous of degree one with respect to x and y because

$$f(\lambda x, \lambda y, z) = z\sqrt{\lambda x \lambda y} = \lambda z\sqrt{xy} = \lambda f(x, y, z).$$

Functions that are homogeneous of degree one are referred to as *linearly homogeneous*, and this property is called *linear homogeneity*. The Swiss mathematician Leonhard Euler proved that linearly homogeneous functions have the property that

$$x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} = f(x, y),$$

which is called *Euler's Rule*.

In economics this special property is used to describe production functions that have constant returns to scale. In other words if an economic output changes by a factor λ when the inputs change by the factor λ, then we say that the production function has constant returns to scale. This is a relatively simple and convenient type of production function.

Linear Homogeneity and Options

In Merton's (1973) classic article on option pricing theory, he demonstrated that the price of a European call option is linearly homogeneous with respect to the stock price and exercise price. Merton derived this result by defining the problem in terms of the rate of return on the option. In this note I take a slightly different but economically equivalent approach.

It is well-known in option pricing theory that the price of an option is obtained by appealing to the risk-neutral or equivalent martingale approach, which changes the probabilities of the expiration value of the underlying stock such that the current stock price is the expected future stock price discounted at the risk-free rate. As such, the expected expiration value of the option is taken using the adjusted probabilities and discounted at the risk-free rate to obtain the well-known Black-Scholes formula.

An option is an asset, however, and by definition the price at time t of any asset is the expected future value of that asset discounted at a rate appropriate for the time value of money and the risk. Consequently, without changing the probability distribution we can at least in principle obtain the option pricing formula by taking the expected expiration value using the actual probabilities and discounting at a rate suitable for the option's risk. In other words,

$$c_t = \exp[-k(T - t)]E[c_T],$$

where k is the risk-adjusted discount rate and T - t is the time to expiration. The above statement holds by definition for any asset, whether an option or not.

The value of a European call option at expiration T is

$$c_T = \text{Max}(0, S_T - X),$$

where S_T is the stock price at expiration and X is the exercise price. Observe that this function is linearly homogeneous with respect to the stock price and exercise price, because

$$\text{Max}(0, \lambda S_T - \lambda X) = \lambda \text{Max}(0, S_T - X).$$

Thus, we know that the expiration value of the option is linearly homogeneous, but we are interested in whether the current value of the option is linearly homogeneous. Here we appeal to the aforementioned result that the current value of any asset is the expected future value discounted back to the current time.

Determining an expected value is a linear mathematical operation. A linear operation performed on a linearly homogeneous function preserves its property of linear homogeneity. Discounting an expected future value, such as by the factor exp[-k(T - t)], is simply multiplying by a constant, which also preserves the function's linear homogeneity. Consequently, the current price of the option is linearly homogeneous with respect to the stock price and exercise price.

The current value of the stock is S_t and the current value of the exercise price is Xexp[-r(T - t)]. Using Euler's Rule, we know that the current option price, which is a function of S_t and t, can be expressed as

$$c_t = S_t\gamma + X \exp[-r(T - t)]\omega,$$

where γ is $\partial c_t/\partial S_t$, ω is $\partial c_t/\partial X$. An interpretation of the above equation is that one call option can be replicated by holding γ units of stock, each worth S_t, and ω units of a risk-free bond with current value Xexp[-r(T - t)].

The total differential of the option price is

$$dc_t = \gamma dS_t + \omega d(X \exp[-r(T - t)]).$$

The differential dS_t can be left in this form. We can, however, obtain the exact differential, dXexp[-r(T - t)]. We simply take the derivative with respect to t,

$$\frac{dX \exp[-r(T - t)]}{dt} = X \exp[-r(T - t)]r.$$

We then obtain the desired differential as

$$dX\exp[-r(T-t)] = X\exp[-r(T-t)]rdt.$$

Consequently,

$$dc_t = \gamma dS_t + \omega X\exp[-r(T-t)]rdt.$$

At this point it is helpful to substitute $c_t - \gamma S_t$ for $\omega X\exp[-r(T - t)]$, leaving us

$$dc_t = \gamma dS_t + (c_t - \gamma S_t)rdt.$$

Itô's Lemma can be used to obtain an equivalent expression for the change in the price of the call,

$$dc_t = \frac{\partial c_t}{\partial S_t}dS_t + \frac{\partial c_t}{\partial t}dt + \frac{1}{2}\frac{\partial^2 c_t}{\partial S_t^{\,2}}S_t^{\,2}\sigma^2 dt,$$

which we can now set equal to the expression above it. By choosing γ to equal $\partial c_t/\partial S_t$, we eliminate the risky term dS_t, leaving the following non-stochastic partial differential equation,

$$c_t r = \frac{\partial c_t}{\partial S_t}rS_t + \frac{\partial c_t}{\partial t} + \frac{1}{2}\frac{\partial^2 c_t}{\partial S_t^{\,2}}S_t^{\,2}\sigma^2,$$

which is the Black-Scholes PDE. The solution, subject to the boundary condition $c_T = \text{Max}(0,S_T - X)$, is of course the Black-Scholes model.

An Application to Forward-Start Options

A forward start option is an option that is purchased today but does not begin until a later date. When the premium is paid today, the purchaser specifies that he wishes to obtain an option at a specified later date that has a certain degree of moneyness. Let time t be the day the forward start option is initiated and j be the date on which the underlying option is received. That is, at time j ($t < j < T$) when the stock price is S_j the option will be granted with an exercise price of αS_j. Thus, an at-the-money option has $\alpha = 1$. An option with exercise price 5% higher than the stock price has $\alpha = 1.05$. Because we do not know today what the stock price will be at j, we cannot prespecify the exercise price. This would appear to make it difficult to price the option, but in fact, it is easy to price the option. We can use the principle of linear homogeneity.

At time j, the option is established and has the property of a standard European option, which has a time to expiration of T - j. Consequently, we know that it is linearly homogeneous with respect to the stock price and exercise price. Writing its value at j as $c(S_j,\alpha S_j,T - j)$, we can state that this option's value is equal to

$$c(S_j,\alpha S_j,T - j) = S_j c(1,\alpha,T - j).$$

The value $c(1,\alpha,T - j)$ is known at all times. It is simply the value of an option where the underlying is worth \$1, the exercise price is α and the time to expiration is T - j.

To price the option today, we need to find a combination of instruments that replicates the option, or in other words, produces a value at j equal to that of the option. It should be obvious that holding $c(1,\alpha,T - j)$ units of the stock will produce a value of $c(1,\alpha,T - j)S_j$ at time j. Consequently, the value of the forward-start option today is

$$c(1,\alpha,T - t)S_t,$$

noting that S_t is the price of the stock today. This is the premium that is paid today to obtain an option that, at time j, will have an exercise price of α times the stock price at j. Of course, this value could also be specified as $c(S_t,\alpha S_t,T-t)$.

If instead we specified today a fixed exercise price of X, then at j we are awarded an option with an exercise price of X and a time to expiration of T - j. Such an option is trivially equal to purchasing an option today with a time to expiration of T - t and an exercise price of X. If the premium is paid today and the option is received later, it is equivalent to just receiving the option today.

It is important, however, to contrast these types of options with forward contracts on options. A forward contract on an option is an agreement to purchase an option at a later date. The option will have an exercise price of X and a time to expiration of T – j when granted. Being a forward contract, no money is paid today but a price that will be paid at j is agreed upon by the two parties. It is also simple to determine this option's price. If today we purchase the option with time to expiration T - t and exercise price X, i.e., the option described in the above paragraph, but borrow the premium, then at time j, we shall have replicated the payoff of the forward contract on the option.

In other words, let $c(S_t,X,T - t)$ be the price of a call option today struck at X with time to expiration T - t. Let $F[c(S_t,X,T - t),j]$ be the forward price agreed upon today to purchase the option with current price $c(S_t,X,T - t)$ at time j. The payoff at time j will be

$$c(S_j,X,T - j) - F[c(S_t,X,T - t),j].$$

This payoff can be replicated by purchasing today the call at price $c(S_t,X,T - t)$ and borrowing $F[c(S_t,X,T - t),j]\exp[-r(j - t)]$. At time j, we are holding the option worth $c(S_j,X,T - j)$ and owe

$F[c(S_t,X,T - t),j]$, which combine to replicate the payoff of the forward contract. Since the transaction is a forward contract, it requires no outlay today. Consequently, the value of the position today, $c(S_t,X,T- t) - F[c(S_t,X,T - t),j]\exp[-r(j - t)]$ must equal zero, meaning that

$$F[c(S_t,X,T - t),j] = c(S_t,X,T - t)\exp[r(j - t]).$$

If the option is written such that the exercise price is specified as αS_j, then instead of holding the call worth $c(S_t,X,T - t)$ today, we hold $c(1,\alpha,T - j)$ shares of stock, as determined above in our derivation of the price of a forward-start option. This instrument combines elements of a forward contract on an option and a forward-start option.

References

The principle that the European call price is linearly homogeneous was first demonstrated in

Merton, R. C. "Theory of Rational Option Pricing." *Bell Journal of Economics and Management Science* 4 (Spring, 1973), 141-183.

It was used to derive the exchange option pricing model (See TN98-04) in

Margrabe, W. "The Value of an Option to Exchange One Asset for Another." *The Journal of Finance* 33 (1978), 177-186.

Forward-Start Options are derived in

Rubinstein, M. "Pay Now, Choose Later." 4 *Risk* (February, 1991), pp. 44-47.

A related type of option, the tandem option, is a sequence of forward-start options and is addressed in

Blazenko, G. W., P. B. Boyle, and K. E. Newport. "Valuation of Tandem Options." *Advances in Futures and Options Research* 4 (1990), 39-49.