

TEACHING NOTE 00-02:
THE LOCAL EXPECTATIONS HYPOTHESIS

Version date: August 15, 2008

C:\CLASSES\TEACHING NOTES\TN00-02.WPD

Almost anyone who has studied the term structure of interest rates is familiar with the unbiased expectations hypothesis (UEH). It says that the forward rate is an unbiased predictor of the future spot rate. The question of whether the UEH is true has occupied a central place in research on the term structure, particularly empirical research. There is no clear consensus empirically, but in fact, it is a simple matter to prove that it cannot be true theoretically and almost surely cannot be true in reality.

For example, let S_0 and F_0 be the spot and forward prices for an asset. The forward price is for a contract that expires one period later and requires delivery of the underlying asset. At time 1, the spot price and forward price converge and are equal to S_1 . At time 0, the expected spot price at time 1 is $E(S_1)$, which is by definition the expectation of the forward price. Of course the forward price converges to the spot price at that point.

The unbiased expectations hypothesis implies that $F_0 = E(S_1)$. By definition, today's spot price is the expected future spot price minus the interest opportunity cost of holding the asset minus a risk premium: $S_0 = E(S_T) - S_0(\exp[r] - 1) - \pi$ where r is the one-period interest rate and π is the risk premium. This statement must hold because rational investors require ex ante compensation for the opportunity cost of funds tied up in the risky asset as well as compensation for the assumption of risk.

It is a well-known fact that to prevent arbitrage, a forward price must equal the spot price compounded at the risk-free interest rate, $F_0 = S_0 \exp[r]$. Substituting $F_{\exp[-r]}$ into the above equation for the spot price and solving for F_0 , we obtain:

$$F_0 = E(S_T) - \pi.$$

In other words, the forward price is the expected spot price minus a risk premium. The presence of the risk premium makes the forward price a biased estimate of the expected spot price, thereby invalidating the unbiased expectations hypothesis.

For further proof, let us suppose that the forward price is unbiased. By definition, the spot price is the expected future spot price discounted at a risky rate, which we shall call k . That is, $S_0 = E(S_T)\exp[-k]$. Substituting F_0 for $E(S_T)$, we obtain the result that $S_0 = F_0\exp[-k]$. But we know that to prevent arbitrage, we must have $S_0 = F_0\exp[-r]$. This implies that $k = r$. Thus, the risky discount rate would be the risk-free rate. Moreover, this would, therefore, be true for any asset. So, if the unbiased expectations hypothesis were true, the risky discount rate would be the same for every asset and equal to the risk-free rate. Except in the uninteresting cases of no risk or investors being risk neutral, this result obviously cannot hold.

A similar but more rigorous hypothesis is the local expectations hypothesis (LEH). It is a result of a term structure that does not permit investors to trade bonds and create arbitrage profits. In fact the LEH is equivalent to the absence of arbitrage. More formally the LEH says that

In the absence of arbitrage opportunities, the forward price of a zero coupon bond will equal the expected spot price if the expected spot price is taken under the equivalent martingale probabilities; these are the artificial probabilities whose existence is guaranteed if there are no arbitrage opportunities in the bond market. The equivalence of the forward price and expected spot price is, however, true only for one-period-ahead forward prices. In other words, the forward price of an n -period bond equals the expected spot price only one period ahead but no further.

The expected returns, taken using the martingale probabilities, of any strategies involving any bonds of any maturity, are equivalent and equal to the one-period spot rate, i.e., the shortest interest rate in the market.

We shall demonstrate the LEH using a simple binomial model. The extension to a more complex continuous time model is more difficult, and we refer the reader to references listed at the end.

We shall use continuously compounded rates, but the proof can be easily restructured for simple rates. For our notation, let $B(a,b)$ be the price at time a of a zero coupon bond maturing at time b . Let $r(a,b)$ be the rate observed on a bond initiated at time a and maturing at time b . When a

= 0, this is a spot bond and a spot rate. For example, we shall have prices and rates like the following:

$B(0,1)$ = price at time 0 of a bond maturing at time 1; this equals $\exp[-r(0,1)]$

$B(0,2)$ = price at time 0 of a bond maturing at time 2; this equals $\exp[-2r(0,2)]$

$B(1,1)$ = price at time 1 of a bond maturing at time 1; this equals 1

$B(1,2)$ = price at time 1 of a bond maturing at time 2; this equals $\exp[-r(1,2)]$.

Let us begin by introducing uncertainty into the market. We assume that the one-period rate, $r(0,1)$ can go up to $r(1,2)^+$ or down to $r(1,2)^-$. Thus, one period later, the price of a one-period bond will be

$$B(1,2)^+ = \exp[-r(1,2)^+], \text{ or}$$

$$B(1,2)^- = \exp[-r(1,2)^-].$$

An arbitrage profit would be possible if we could create a self-financing strategy with a guaranteed non-negative profit. There are two possible arbitrage strategies: (1) buy a one-period bond, financing it by selling a two-period bond or (2) buy a two-period bond, financing it by selling a one-period bond, liquidating the position at time 1.

In the first strategy, an arbitrage profit would occur if $\exp[r(0,1)] > B(1,2)^-/B(0,2)$. This statement simply means that if we buy a one-period bond we earn $r(0,1)$. The expression $B(1,2)^-/B(0,2)$ is the one plus the rate of return of the two-period bond if rates go down. This is our financing cost, and the case of rates falling provides the worst case outcome. That is, we short the two-period bond and rates fall leading to an increase in its price. If we still manage to earn more off of our one-period bond than the worst case financing outcome, we have earned an arbitrage profit.

In the second strategy, an arbitrage profit would occur if $\exp[r(0,1)] < B(1,2)^+/B(0,2)$. The rate $r(0,1)$ is our financing cost. The expression $B(1,2)^+/B(0,2)$ is one plus the return on our two-period bond in the worst outcome, that of rates increasing. If that return exceeds the financing cost, we have earned an arbitrage profit.

Thus, to prevent arbitrage, neither of these conditions must occur. Thus, we must have

$$B(1,2)^+/B(0,2) < \exp[r(0,1)] < B(1,2)^-/B(0,2).$$

This statement implies that there is a set of weights, which we shall call p and $1 - p$, such that

$$p B(1,2)^+/B(0,2) + (1 - p)B(1,2)^-/B(0,2) = \exp[r(0,1)].$$

The left-hand side can be viewed as the expected return on a two-period bond held for one period and this equals the right-hand side, which is the one-period rate. To interpret this as an expected value, we must recognize these weights as risk neutral or equivalent martingale weights. We call them risk neutral because the above expression is the expected return that a risk neutral investor would have. These are, therefore, the probabilities that would be used if current market prices held and investors were risk neutral. They are also called equivalent martingale probabilities, because they convert the discounted bond price into a martingale, which is a stochastic process with zero expected return. In the above equation, a martingale is obtained by discounting at the one period rate, i.e., multiplying by $\exp[-r(0,1)]$. The left-hand side is the discounted bond price and the right-hand side is 1.0, thereby implying a zero expected return.

To generalize the above result, let the two-period bond be a bond of any maturity, say n . Then repeat the above exercise and you will easily see that it holds. A more formal proof can demonstrate that there is no bond trading strategy that permits arbitrage.

The result that the expected return, under the martingale probability is the one-period rate does not apply to a strategy that spans more than one period. One reason for this is that arbitrage is a risk-free strategy and there is no true risk-free rate beyond one period. While indeed buying a zero coupon bond of maturity n and holding it to maturity leads to a risk-free return, arbitrage strategies require a rebalancing of positions to properly offset the changing risk of the component instruments. This rebalancing requires trading in intermediate periods. Such trading requires buying and selling, which exposes one to the uncertainties of interest rate changes, thereby eliminating the notion of a risk-free rate.

We have now demonstrated that the expected return, using the martingale probabilities equals the one-period rate. Next we demonstrate the other implication of the LEH, that the expected price of the two-period bond at time 1 is the forward rate.

Let us express the above equation as

$$pB(1,2)^+/B(0,2) + (1 - p)B(1,2)^-/B(0,2) = 1/B(0,1),$$

noting that $1/B(0,1)$ is $\exp[r(0,1)]$. Multiply both sides by $B(0,2)$, we obtain

$$pB(1,2)^+ + (1 - p)B(1,2)^- = B(0,2)/B(0,1).$$

By definition, the right-hand side is the forward price. The left-hand side is the expected spot price under the martingale probabilities. So the forward price is the expected spot price under the

martingale probabilities. In addition it generalizes to bonds of any maturity, but again, applies only to one period ahead.

Finally, let us show that the equivalence of the forward price to the expected spot price does not imply equivalence of the forward rate to the expected spot rate.

The right-hand side can be written as $\exp[-r(0,2) + r(0,1)]$. Taking logs gives us the forward rate, which, as is commonly known, in continuous time is just the difference in the spot rates for the respective maturities. If we take the log of the right-hand side, we must take the log of the left-hand side, but it does not reduce to $pr(1,2)^+ + (1 - p)r(1,2)^-$, the expected spot rate. It is simply $\log[pB(1,2)^+ + (1 - p)B(1,2)^-]$. While this will be close to the expected spot rate, it is not equivalent.

Our academic finance profession has done a great disservice in teaching the so-called unbiased expectations hypothesis. We have left practitioners with the belief that forward interest and exchange rates tell the direction of future spot rates. Hopefully it will take no more than one more generation to correct this drastic mistake.

References

The Local Expectations Hypothesis is developed in continuous time in

Cox, J. C., J. E. Ingersoll, Jr. and S. A. Ross, "A Re-Examination of Traditional Hypotheses About the Term Structure of Interest Rates." *The Journal of Finance* 36 (1981), 769-799.

See also:

Longstaff, F. A. "Arbitrage and the Expectations Hypothesis." *The Journal of Finance* 60 (2000), 989-994.

D. M. Chance and D. Rich. "The False Teachings of the Unbiased Expectations Hypothesis." *Journal of Portfolio Management* 27 (Summer, 2001), 83-95.

A good textbook treatment is:

Jarrow, R. A. *Modeling Fixed Income Securities and Interest Rate Options*, 2nd ed. Stanford, CA: Stanford University Press (2002). See Chapter 5. For an excellent treatment that generalizes the no-arbitrage rule to all bond trading strategies, see Chapter 6.

The general theorem of no-arbitrage trading is found in:

Harrison, J. M. and D. M. Kreps. "Martingales and Arbitrage in Multiperiod Securities Markets."
Journal of Economic Theory 20 (1970), 381-408.