

**TEACHING NOTE 00-07:
THE REFLECTION PRINCIPLE IN FINANCE**

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In financial applications of stochastic processes one is often interested in the probability that a random variable has reached a certain level by a certain time. Sometimes additional conditions are imposed. The most common situation is in the valuation of barrier options. Such options specify a critical level, called a barrier. If the asset price hits the barrier level, the option either terminates or activates. The former are called *out-options*, sometimes *knock-out options*, and the latter are called *in-options*, sometimes *knock-in options*. To price these options we need to model the condition of whether the barrier is hit or not, as well as the condition of where the asset price is at expiration relative to the exercise price, given whether the barrier was hit or not prior to expiration. In order to understand the pricing of barrier options, there are a number of basic results that we must establish. We start with the simplest situation of all and then build to progressively more difficult but more realistic settings.

Let $X(t)$ be a random variable following some stochastic process. Suppose we wish to know $\text{Prob}(\max_{0 \leq t \leq T} X(t) \geq a)$, $0 \leq t \leq T$. This is not the probability that $X(t) \geq a$, which is that the value of X at a specific time t is at least a , but rather the probability that the maximum value achieved by $X(t)$ during the period from 0 to T is at least a .¹ Sometimes we might also wish to know something like this: $\text{Prob}(X(T) \geq a, \max_{0 \leq t \leq T} X(t) < b)$, $a < b$, $0 \leq t \leq T$. Here we are looking for the probability that X at time T is at least the value of a , but over the period 0 to T , $X(t)$ has not hit the level b . These probabilities can be obtained using a procedure called the *reflection principle*.

The Reflection Principle for a Discrete Time Stochastic Process

We illustrate the reflection principle first for a simple binomial process. Let $X(t) = 1$ or -1 , $t = 1, 2, 3, 4$. Thus, we have a four-period binomial process. Define a random

¹In mathematics, the maximum value is often called the *supremum*, and the minimum value is often called the *infimum*. Here, however, we shall use the terms *max* and *min*.

variable $Y(t) = \sum_{t=1}^4 X(t)$. $Y(t)$ is simply the sum of $X(t)$ and is between -4 and +4 at time

4. The set of possible outcomes and values for $Y(t)$ is listed below.

Outcome	Time 1	Time 2	Time 3	Time 4	Maximum	Minimum
1	1	2	3	4	4	1
2	1	0	1	2	2	0
3	1	2	1	2	2	1
4	1	2	3	2	3	1
5	-1	0	1	2	2	-1
6	1	2	1	0	2	0
7	1	0	-1	0	1	-1
8	1	0	1	0	1	0
9	-1	-2	-1	0	0	-2
10	-1	0	-1	0	0	-1
11	-1	0	1	0	1	-1
12	-1	0	-1	-2	0	-2
13	-1	-2	-1	-2	-1	-2
14	-1	-2	-3	-2	-1	-3
15	1	0	-1	-2	1	-2
16	-1	-2	-3	-4	-1	-4

Suppose we wish to know the following probabilities, which can be obtained by simply counting the outcomes:

$$\text{Prob}(\max Y(t) \geq 3) = 2/16$$

$$\text{Prob}(\max Y(t) \geq 2) = 6/16$$

$$\text{Prob}(\max Y(t) \geq 1) = 10/16.$$

Alternatively, we can observe that the probabilities are obtained by applying the formula:

$$\text{Prob}(\max Y(t) \geq a) = 2\text{Prob}(Y(t) \geq a) - \text{Prob}(Y(t) = a),$$

which for the above problems can be used in the following manner:

$$\text{Prob}(\max Y(t) \geq 3) = 2\text{Prob}(Y(4) \geq 3) - \text{Prob}(Y(4) = 3) = 2(1/16) - 0/16 = 2/16$$

$$\text{Prob}(\max Y(t) \geq 2) = 2\text{Prob}(Y(4) \geq 2) - \text{Prob}(Y(4) = 2) = 2(5/16) - 4/16 = 6/16$$

$$\text{Prob}(\max Y(t) \geq 1) = 2\text{Prob}(Y(4) \geq 1) - \text{Prob}(Y(4) = 1) = 2(5/16) - 0/16 = 10/16.$$

There is nothing special about time 4; the rule would also apply to any time prior to 4.

Although it is easy to count the outcomes and determine the probability, in continuous time problems there are an infinite number of outcomes. We need a tool to calculate the probability directly. That tool is the reflection principle. Let us focus on the second example above, $\text{Prob}(\max Y(t) \geq 2)$, for which we observe there to be 6 outcomes that satisfy the criterion of Y reaching at least a value of 2 at any time prior to and including time 4. The reflection principle says that this probability is the same as two times as the probability that $Y(4)$ is at least 2 at time 4 minus the probability that $Y(4)$ is precisely 2 at time 4. Let us consider a case where Y hits a level of 2 before time 4. This occurs in outcomes 1, 3, 4, and 6. When Y reaches 2 at say time 2, it can then move on up and be as high as 4 at time 4 or can fall back down to as low as 0. In each of those cases we can match the path taken in one outcome with an exactly opposite path. For example, notice the following values for Y :

$$\text{Path 1 : } Y(t) = 2, 3, 4 \text{ as } t = 2, 3, 4$$

$$\text{Path 6: } Y(t) = 2, 1, 0 \text{ as } t = 2, 3, 4$$

The above paths are a mirror image of each other and both qualify as having hit 2 by time 4. The following paths are similarly characterized:

$$\text{Path 3: } Y(t) = 2, 1, 2 \text{ as } t = 2, 3, 4$$

$$\text{Path 4: } Y(t) = 2, 3, 2 \text{ as } t = 2, 3, 4.$$

Let us consider the other mirror image path in which Y hits 2 or above by time 4. At time 3, we have

$$\text{Path 1: } Y(t) = 3, 4 \text{ as } t = 3, 4$$

$$\text{Path 4: } Y(t) = 3, 2 \text{ as } t = 3, 4,$$

These cases, paths 1, 3, 4 and 6, however, overstate the case as paths 1 and 4 are repeated. We correct for that later. Thus, so far we have accounted for 4 of the 6 possibilities. The other cases are outcomes 2 and 5 in which $Y(t)$ hits 2 at time 4.

We are interested in the set of paths in which $Y(t)$ hits 2 by time 4. Regardless of what happens to $Y(t)$ after it hits 2 for the first time, such a path should be included in our

overall total for purposes of computing the probability. As we showed above for every one of these paths in which $Y(t)$ hits 2 by time 4, there is an opposite (reflected) path that should also be counted. If $Y(4)$ is at 2 or above by time 4, it either hit 2 prior to time 4 or hits 2 at time 4. Thus, if we double the probability that $Y(4)$ is at 2 or above at time 4, we have included those paths in which $Y(4)$ hit 2 at time 4 for the first time as well as those paths in which $Y(t)$ hit 2 prior to time 4 and stayed above 2 or fell below. For if $Y(t)$ hit 2 prior to time 4 and stayed below, there was an opposite path in which $Y(t)$ hit 2 prior to time 3 and ended up above 2. We have, however, double counted some outcomes since $Y(t)$ could hit 2 before time 4 and ended up exactly at 2 at time 4, but we have already included those cases where $Y(t)$ hit 2 for the first time at time 4. Thus, we need to subtract the probability that $Y(4) = 2$. Thus, we have

$$\text{Prob}(\max Y(t) \geq 2) = 2\text{Prob}(Y(4) \geq 2) - \text{Prob}(Y(4) = 2).$$

In general we have

$$\text{Prob}(\max Y(t) \geq a) = 2\text{Prob}(Y(T) \geq a) - \text{Prob}(Y(T) = a).$$

In the above examples, we were looking at the probability that a random variable exceeded a value at some time during a period of time. What if we were interested in the probability that the random variable *did not exceed* a particular value during a period of time. Using the law of total probability, we can say that

$$\text{Prob}(\max Y(t) < a) + \text{Prob}(\max Y(t) \geq a) = 1.$$

Therefore,

$$\text{Prob}(\max Y(t) < a) = 1 - \text{Prob}(\max Y(t) \geq a).$$

So by knowing the probability of the maximum exceeding a given value, a result we have previously derived, we easily obtain the probability of the maximum not exceeding a given value. Note that if we wanted the statement, $\text{Prob}(\max Y(t) \leq a)$, we would use $1 - \text{Prob}(\max Y(t) > a)$.

This approach, as noted earlier, is called the reflection principle. It is also called the *method of images*, because it works as though a mirror were placed on the plane at the point at which the critical level is hit, and the mirror image of the ensuing path is captured.

Now let us examine the problem focusing on the minimum. For example, we might be interested in the probability that the random variable is less than or equal to a given value at some time during a period of time. In this case the result is

$$\text{Prob}(\min Y(t) \leq a) = 2\text{Prob}(Y(T) \leq a) - \text{Prob}(Y(T) = a).$$

Likewise, if we are interested in the probability that $\min Y(t)$ is greater than a,

$$\text{Prob}(\min Y(t) > a) = 1 - \text{Prob}(\min Y(t) \leq a).$$

Notice that since our outcomes are discrete, we have to allow for the possibility that $Y(T) = a$ for some outcome. This creates a little confusion for the case where we do not want Y to exactly equal a . Suppose, for example, that instead of our original problem, $\text{Prob}(\max Y(t) \geq a)$, we want $\text{Prob}(\max Y(t) > a)$. In this case, we can re-specify our problem in terms of the next highest discrete value, such as $a + \Delta = b$. Then we have $\text{Prob}(\max Y(t) > a) = \text{Prob}(\max Y(t) \geq b)$. As we show in a later section, if our random variable is continuous, then we can ignore the event $Y(T) = a$, since this has zero probability.

Let us now summarize our results and write them more generally. We assume that time starts at time 1 and ends at time T :²

$$\text{Prob}(\max Y(t) \geq a) = 2\text{Prob}(Y(T) \geq a) - \text{Prob}(Y(T) = a)$$

$$\text{Prob}(\max Y(t) < a) = 1 - \text{Prob}(\max Y(t) \geq a)$$

$$\text{Prob}(\min Y(t) \leq a) = 2\text{Prob}(Y(T) \leq a) - \text{Prob}(Y(T) = a)$$

$$\text{Prob}(\min Y(t) > a) = 1 - \text{Prob}(\min Y(t) \leq a).$$

What we have done so far is be able to make probability statements about whether a particular level has been hit or not by a certain time. In terms of a barrier option, this might correspond to answering the question of what is the probability that the barrier has been hit. Sometimes we are interested in joint outcomes, for example $\text{Prob}(Y(4) \geq a, \min Y(t) \leq b)$. Such a specification enables us to ultimately examine payoffs for barrier options, which are contingent on both the asset price at expiration being above or below a given level *and* the barrier either being hit or not hit by expiration.

For the specification $\text{Prob}(Y(4) \geq a, \min Y(t) \leq b)$, we want to know whether the final value is at least equal to a and that somewhere during the period of time 1 to time 4,

²It would be a worthwhile exercise to check these formulas using the example presented above.

the value of $Y(t)$ hit or fell below the level b . Let us work a problem. Let $a = 0$ and $b = -1$. To formally state the problem, we want

$$\text{Prob}(Y(4) \geq 0, \min Y(t) \leq -1).$$

Looking at the outcomes, we see the following: There are five outcomes in which this condition is met (outcomes 5, 7, 9, 10, and 11) so the probability is $5/16$. Let us see how we can use the reflection principle to obtain that result. Let Y travel down to -1 . Now we need it to travel up to 0 . The probability of that occurring is the same as the probability of it traveling down to -1 and then down from -1 to -2 . So we need the probability of $Y(4) \leq -2$. We see that there are five such outcomes so the probability is $5/16$.

Consider another example in which we observe Y move upward. Let us say we want

$$\text{Prob}(Y(4) \leq 0, \max Y(t) \geq 1).$$

We observe that there are five such outcomes that meet this requirement (outcomes 6, 7, 8, 11, and 15). So the probability is $5/16$. We can obtain that value using the reflection principle. We note that if Y travels up to $+1$, we then need it to travel down to 0 . The probability of that occurring is the same as it traveling up to $+1$ and then up to $+2$. Thus, we need $\text{Prob}(Y(4) \geq 2)$, which is $5/16$.

All of the remaining possibilities can be similarly obtained. Note, however, that many outcomes, such as hitting $+2$ and then falling to -3 , are not possible. We can summarize these results by saying that in general,

$$\text{Prob}(Y(T) \leq b, Y(t) \geq a) = \text{Prob}(Y(T) \geq 2a - b)$$

$$\text{Prob}(Y(T) \geq b, Y(t) \leq a) = \text{Prob}(Y(T) \leq 2a - b).$$

The Reflection Principle for a Continuous Stochastic Process

The continuous analog to the above example uses the process known as Brownian motion. We assume the reader is familiar with the concept of Brownian motion. We illustrate the reflection principle for the case of Brownian motion with no drift and with local volatility of σ . The reflection principle cannot be directly applied when there is a drift, but we shall show the necessary adjustment. Also, there are a tremendous number of variations of the example problem we use below and these are worthwhile to practice on.

Suppose the Brownian motion process is the random variable Z , which evolves over the time interval from 0 to T . We start at $Z(0) = 0$ and ultimately end at $Z(T)$. We assume that there is no drift, although we show how that assumption is relaxed later. We do assume a volatility of σ . Consequently, $Z(T)$ has a variance of $\sigma^2 T$.

The results for the probability of the maxima and minima are adapted in a straightforward manner from those above for the discrete case.

$$\text{Prob}(\max Y(t) \geq a) = 2\text{Prob}(Y(T) \geq a)$$

$$\text{Prob}(\max Y(t) < a) = 1 - 2\text{Prob}(Y(T) \geq a)$$

$$\text{Prob}(\min Y(t) \leq a) = 2\text{Prob}(Y(T) \leq a)$$

$$\text{Prob}(\min Y(t) > a) = 1 - 2\text{Prob}(\min Y(T) \leq a)$$

Note that the only difference from the discrete case is that we do not have to subtract the probability of $Y(T)$ being exactly equal to a since that probability is zero.

If we are interested in the joint probabilities, we cannot simply count outcomes as we did in the discrete case since there are an infinite number. We must appeal to the reflection principle. There are a wide variety of problems that could be worked here. We shall take a look at a good representative problem, which is fairly complex.

Suppose we have $Z_0 (= 0) > a > b$. We are interested in the following probability:

$$\text{Prob}(Z(T) > a, \min Z(t) > b).$$

In other words, we want $Z(t)$ to not fall below b and to end up above a . First note that

$$\text{Prob}(Z(T) > a, \min Z(t) > b) + \text{Prob}(Z(T) > a, \min Z(t) \leq b) = \text{Prob}(Z(T) > a).$$

Thus, our problem can be stated as

$$\text{Prob}(Z(T) > a, \min Z(t) > b) = \text{Prob}(Z(T) > a) - \text{Prob}(Z(T) > a, \min Z(t) \leq b).$$

The first probability on the right-hand side is simple; it is a standard normal. So now we must solve the problem $\text{Prob}(Z(T) > a, \min Z(t) \leq b)$.

This problem has $Z(t)$ fall to b and then bounce back up to end up above a . The probability of this occurring is the same as the probability of $Z(t)$ falling to b and then falling a distance $(a - b)$. So the total distance it travels is $b - (a - b) = 2b - a$. So

$$\text{Prob}(Z(T) > a, \min Z(t) \leq b) = \text{Prob}(Z(T) < 2b - a).$$

This is a standard normal and is given by

$$\text{Prob}(Z(T) < 2b - a) = N\left(\frac{2b - a}{\sigma\sqrt{T}}\right).$$

Going back to our original problem:

$$\begin{aligned} \text{Prob}(Z(T) > a, \min Z(t) > b) &= \text{Prob}(Z(T) > a) - \text{Prob}(Z(T) > a, \min Z(t) \leq b) \\ &= \text{Prob}(Z(T) > a) - N\left(\frac{2b - a}{\sigma\sqrt{T}}\right) \\ &= 1 - N\left(\frac{a}{\sigma\sqrt{T}}\right) - N\left(\frac{2b - a}{\sigma\sqrt{T}}\right). \end{aligned}$$

There are numerous other problems of this sort that can be specified. We can change the position of a and b with respect to each other and with respect to $Z(0)$. We can change the minimum to the maximum, meaning that we hit or do not hit the barrier from below. One particularly important situation is when the asset has a drift. If that is the case, we must multiply the probability obtained by the reflection principle by $\exp(2b\mu/\sigma^2)$ where b is the barrier and μ is the drift. In addition all of the statements within the normal probability functions above have implicitly the subtraction of a zero drift in the numerator. If the drift is not zero, we must subtract it. Thus, the probability above would be

$$\text{Prob}(Z(T) > a, \min Z(t) > b) = 1 - N\left(\frac{a - \mu T}{\sigma\sqrt{T}}\right) - \exp(2b\mu/\sigma^2) N\left(\frac{2b - a - \mu T}{\sigma\sqrt{T}}\right).$$

Note that the drift adjustment in the numerator of the normal probability argument is made to the first probability, but it is not obtained using the reflection principle so we do not have to multiply it by the exponential expression.

What we have done so far gives us the tools to enable us to evaluate the kinds of payoffs provided by barrier options. But, barrier options are not written on a simple Brownian motion; they are written on an asset.

Application to a Lognormally Distributed Asset Return

In most finance applications, we assume that there is an asset that follows a Geometric Brownian motion process, which implies that its return is lognormally distributed. In order to use the results above, we have to model the log return. We start with an asset priced at S . At time T , the asset price is $S(T)$. We assume the usual condition that the asset return is given by the familiar stochastic differential equation,

$$dS/S = \mu dt + \sigma dZ,$$

where Z is a standard Brownian motion, as used above.³ The relationship between the current price and the terminal price is

$$S(T) = S \exp[x].$$

Since $\log[S(T)/S] = x$, then x is, therefore, the log return over the period spanned by T . It is well-known that under these assumptions, x is lognormally distributed with mean μT and variance $\sigma^2 T$.

Suppose we are interested in the condition that $S(T)$ is below a barrier H , which lies below the current value of S . Then

$$\begin{aligned} S(T) &< H \\ \Rightarrow S \exp[x] &< H \\ \Rightarrow x &< \log(H/S). \end{aligned}$$

Thus, we can specify our models such that a barrier being hit is written in terms of the relationship of x to $\log(H/S)$. Likewise, the condition of, for example, a call option expiring in the money, $S(T) > K$, is specified as $x > \log(K/S)$. In fact this is the specification used in deriving the Black-Scholes model for standard European options. Thus, finding the probability of $x > \log(K/S)$ involves integrating over the range of x from $\log(K/S)$ to ∞ .

Finding the probabilities of a barrier being hit or not or that of an option being in-the-money at expiration as well as the barrier being hit or not are a straightforward application of principles presented earlier in this document, with appropriate adjustments to reflect the fact that we model the log return as discussed in this section. Since it is an asset we are dealing with, there is inevitably a drift, so the exponential term for the drift adjustment must be made. Also, the drift of x must be subtracted in the numerator of the argument of the normal probability.⁴ Another factor that must be considered is that the pricing of options uses the risk neutral/equivalent martingale approach, which means that the drift is replaced by the risk-free rate.

³In some of the above examples, we used a Brownian motion with a volatility of σ and in some cases, we imposed a drift of μ . At this point, we assume the case of zero drift and unit volatility for Z .

⁴Again, multiplying by the exponential term is necessary only when the reflection principle is used to obtain a probability; however, the adjustment in the numerator of the normal probability argument is necessary for any probability specification in which the underlying random variable has a non-zero drift. This is simply the standard normal specification, (value - mean)/standard deviation.

Let us illustrate one example of these points by examining the probability that a barrier is hit by a future time, which might be something like the expiration of an option. With the barrier below the current asset price, and the future time being $T \geq t$, what we want is

$$\text{Prob}(S(t) \leq H) \text{ for some } t.$$

This is written alternatively as,

$$\text{Prob}(\min S(t) \leq H).$$

To more thoroughly illustrate this problem, let us solve the case for an analogous problem using the simple Brownian motion Z . Specifically, we would first like to know whether $Z(t)$ hits a barrier, b , from above, before time T :

$$\text{Prob}(Z(t) < b) \text{ for some } t \leq T = \text{Prob}(\min Z(t) < b).$$

We have already seen that this is

$$2\text{Prob}(Z(T) \leq b).$$

Thus, for the zero drift case, this is

$$2N\left(\frac{b}{\sigma\sqrt{T}}\right).$$

The procedure for finding this result, however, would require that we use the reflection principle. Consequently, there would be the drift adjustment previously mentioned if the drift is not zero.

If the underlying is our asset, the problem is set up as follows. We want $\text{Prob}(S(t) \leq H)$ for some t . In other words, we do want $S(t)$ to fall below H at some time prior to T . One measure that gives us this probability is

$$1 - \text{Prob}(S(T) > H, \min S(t) > H).$$

That is, $\text{Prob}(S(T) > H, \min S(t) > H)$ establishes the case where $S(t)$ does not go through the barrier and ends up above the barrier. This is definitively the case of $S(t)$ not hitting the barrier. Thus, $1 -$ that probability is the probability of $S(t)$ hitting the barrier. We can then express this as

$$\begin{aligned}
& 1 - \text{Prob}(S(T) > H, \min S(t) > H) \\
&= 1 - (\text{Prob}(S(T) > H) - \text{Prob}(S(T) > H, \min S(t) < H)) \\
&= 1 - \text{Prob}(S(T) > H) + \text{Prob}(S(T) > H, \min S(t) < H) \\
&= \text{Prob}(S(T) < H) + \text{Prob}(S(T) > H, \min S(t) < H).
\end{aligned}$$

The first probability is easy. It is standard normal:

$$\text{Prob}(S(T) < H) = 1 - N\left(\frac{\ln(S/H) + \mu T}{\sigma\sqrt{T}}\right).$$

Note the similarity to the Black-Scholes formula. If H were the exercise price, the above would be analogous to $1 - N(d_2)$, or the probability that the call would not be exercised. Note, however, the drift term, μT . In the Black-Scholes model, this term is recognized as $r - \sigma^2/2$.⁵ But this is the drift of the log return and would be the same here.

The second probability is obtained by the drift-adjusted reflection principle:

$$\text{Prob}(S(T) > H, \min S(t) < H) = (H/S)^{2\mu/\sigma^2} N\left(\frac{\ln(H/S) + \mu T}{\sigma\sqrt{T}}\right).$$

Note that we have substituted $(H/S)^{2\mu/\sigma^2}$ for $\exp[2\mu \log(H/S)/\sigma^2]$.

If the asset drift were zero, these expressions would simplify to the result we obtained when specifying the probability for Z hitting the barrier.

First Passage Time

Another useful result is the first passage time density. The time it takes until the barrier is hit is a random variable. We can find the density of that random variable. This will be useful in analyzing one feature of some barrier options, a rebate paid if a knock-out option hits the barrier or if a knock-in option never hits the barrier. Let us initially go back to the simple Brownian motion case.

Let us define τ_b to be the first time that the random variable $Z(t)$ hits the barrier b . We look at the case in which the barrier is above the starting level. We consider time t on a continuum from 0 to T . For every time point $t \leq T$, we want to know the probability that the barrier has been hit. Thus, we want to know $\text{Prob}(\tau_b \leq t)$ for any t . From previous results, we know this would be

⁵Here r is the continuously compounded interest rate.

$$\begin{aligned}
\text{Prob}(\tau_b \leq t) &= \text{Prob}(\max Z(i) > b) \\
&= 2 \text{Prob}(Z(t) > b) = 2 \left(\frac{1}{\sqrt{2\pi t}} \int_b^\infty \exp\left[-\frac{1}{2} \frac{x^2}{t}\right] dx \right) \\
&= \sqrt{\frac{2}{\pi t}} \int_b^\infty \exp\left[-\frac{1}{2} \frac{x^2}{t}\right] dx.
\end{aligned}$$

At this point it is best to change our variable from x to $y\sqrt{t}$. This means that $x^2 = y^2 t$ so $x^2/t = y^2$. Also, we have to adjust the lower limit of integration, which will become b/\sqrt{t} . Also since $y = x/\sqrt{t}$, then the variance of y is the variance of x times $1/t$. But the variance of x is t so the variance of y is simply 1. Then 1 replaces the t in the denominator under the square root sign. Thus,

$$\text{Prob}(\tau_b \leq t) = \sqrt{\frac{2}{\pi}} \int_{b/\sqrt{t}}^\infty \exp\left[-\frac{1}{2} y^2\right] dy.$$

Differentiation of this expression with respect to t gives the density function:

$$\frac{d \text{Prob}(\tau_b \leq t)}{dt} = \frac{bt^{-3/2}}{\sqrt{2\pi}} \exp\left[-\frac{1}{2} \frac{b^2}{t}\right]$$

If we were approaching the barrier from above, the a would be $-a$. When there is a drift of μ and a volatility of σ , this is

$$\frac{bt^{-3/2}}{\sigma\sqrt{2\pi}} \exp\left[-\frac{1}{2} \left(\frac{b - \mu t}{\sigma\sqrt{t}}\right)^2\right].$$

References

There are numerous references on the reflection principle in stochastic processes. Two of the best are

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