

**TN05-01:
THE PRICING AND INTEREST SENSITIVITY OF
FLOATING-RATE SECURITIES**

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Many companies issue securities in which the coupons adjust according to a specified interest rate observable in the financial market. These instruments are called *floating-rate securities*. Oftentimes the original maturities of the securities are in the range of one to ten years and, hence, are usually called *floating-rate notes* and sometimes *floaters*.¹ These instruments have somewhat different characteristics from other fixed-income securities, and hence, require a separate look.²

Define the issue date as time 0. At this time, the first coupon is set to c_0 . On each date 1, 2, ..., T-1, the coupon will be reset to c_1, c_2, \dots, c_{T-1} , respectively. The coupons are paid at dates 1, 2, ..., T-1, and T. Thus, the coupon is set at the beginning of a period and paid at the end. This procedure assures that a borrower will know at any time period at what rate interest is accruing. There are other schemes possible, including the setting of the rate at the end of the period, but we consider only the conventional specification in this teaching note.

The coupon is set according to a formula that relates it to a market rate. We use the simple specification that the coupon rate equals the market rate. Most floating-rate securities have the coupon set at a spread over the market rate. This spread reflects the default risk of the borrower over the default risk reflected by the market rate. This teaching note assumes no spread. Other coupon-setting schemes are briefly mentioned later.

Let the face value be 1. The annualized discount rate used to find the present value of the payments is $r_{j,j+1}$, where j is any time during the life of the bond, depending on where we are when we are valuing the bond. Thus, if we are discounting a payment

¹On occasion we hear the term *variable rate* used instead of floating rate. This practice is more common in the mortgage industry, where variable rate mortgages are common. Variable rate mortgages are quite different from floating rate securities in that mortgage payments include principal, whereas floating rate securities usually have the principal paid at maturity. Also, mortgages are heavily influenced by prepayments of principal.

²One could rightfully argue that these instruments should not even be called *fixed-income securities*, because the income they provide is hardly fixed. But tradition has resulted in using the term for any type of bond.

that occurs in 90 days, we use the current rate for 90-day zero coupon instruments. We start off by structuring this instrument like a Eurodollar loan, whereby discounting is done by using the factor $1 + r_{j,j+1} * (\text{days between } j \text{ and } j+1) / 360$, where *days* is the number of days over which we are discounting. We use the 360-day convention, because that is used in the Eurodollar market.

Now let us find the price of the security at various points in its life.

Pricing the Floating-Rate Security

We are interested in pricing the floating-rate security at two distinct time points: on any date on which the coupon is set and in between the coupon-setting dates.

Pricing the Floating-Rate Security on a Coupon-Setting Date

At maturity, the security pays its face value of 1. Hence, its value at that point is 1. Now step back to time $T-1$. On that date, the coupon rate, $c_{T-1,T}$, is set to $r_{T-1,T}$, the annualized rate at time $T-1$ for the period $T-1$ to T . The actual coupon is $c_{T-1,T} q_{T-1,T}$ where $q_{T-1,T}$ is the number of days between $T-1$ and T divided by 360. To obtain the value of the security at $T-1$, V_{T-1} , we discount the upcoming payments of coupon and principal by the appropriate market discount factor. Since $c_{T-1,T} = r_{T-1,T}$, we have,

$$V_{T-1} = \frac{1 + c_{T-1,T} q_{T-1,T}}{1 + r_{T-1,T} q_{T-1,T}} = 1.$$

Now step back to $T-2$. At that time, the coupon rate, $c_{T-2,T-1}$, is set to $r_{T-2,T-1}$. This coupon will be paid on day $T-1$. So on day $T-2$, we know the upcoming coupon but we do not know the market rate on day $T-1$ that will determine the coupon on day T . We do know, however, that on day $T-1$, the *value* of the final coupon (the one paid on day T) and principal (also paid on day T) will be 1. Hence, all we have to do is discount the known upcoming coupon payment plus 1. This value of 1 captures the value at time $T-1$ of the payments upcoming at T . Thus, the value of the bond at $T-2$ is,

$$V_{T-2} = \frac{1 + c_{T-2,T-1} q_{T-2,T-1}}{1 + r_{T-2,T-1} q_{T-2,T-1}} = 1.$$

Continuing this process by backing up to the present gives a value of 1 at time 0.

Note that the value is 1 at any coupon date. This point captures the essence of a floating-rate security. It is designed for the coupon to adjust to the market interest rate so that the price will not stray far from par value. Indeed, the price will come back to par

value at any coupon date, because at that point, the coupon rate realigns with the discount rate in the market.³

Pricing the Floating-Rate Security Between Coupon-Setting Dates

Now consider the value between coupon-setting dates. Position ourselves at time t , where $j \leq t \leq j+1$ and j and $j+1$ are any two consecutive coupon-setting dates. On day j , the coupon was set at $c_{j,j+1}$ but now, on day t , the discount rate is $r_{t,j+1}$, which reflects the appropriate discount rate for the period t to $j+1$. Again, this is an annualized rate, so we must adjust by the factor $q_{t,j+1}$, which reflects the number of days between t and $j+1$, which is the number of days remaining until the next coupon is paid. Note, however, that the coupon payment is still multiplied by $q_{j,j+1}$, because the coupon is an annualized rate and provides interest to the holder for the period j to $j+1$. We are discounting the coupon over the shorter period t to $j+1$.

The value of the security is

$$V_t = \frac{1 + c_{j,j+1}q_{j,j+1}}{1 + r_{t,j+1}q_{t,j+1}}$$

This value will not, in general, equal 1. Hence, the value of the floating rate security will deviate from its par value. If the length of the period is not too long, however, the value will not deviate far from 1. Note that if we slide t backward to time j or forward to time $j+1$, this formula will converge to 1.

A Numerical Example of the Pricing of a Floating-Rate Security

Consider a floating-rate security maturing in two years, with semiannual coupons set to LIBOR. The four coupons are set on days 0, 180, 360, and 540 and are paid on days 180, 360, 540, and 720, respectively. Suppose on day 540, the rate is set at 5.5%. Then on day 540, the price is

$$V_{540} = \frac{1 + 0.055(180 / 360)}{1 + 0.055(180 / 360)} = 1.$$

As we noted, the price always goes back to par on a coupon date. Now let us move to day 600, which is 60 days after the third coupon and 120 days before the last coupon (and

³In practice, the spread for default risk can keep the coupon rate and the market discount rate from realigning, because the default risk can change, resulting in a different spread in the market discount rate than the spread specified in the formula that determines the coupon.

principal repayment). The coupon was set on day 540 at 5.5%. Now assume that on day 600, the annualized rate for 120 days is 5.9%. The value of the security is

$$V_{540} = \frac{1 + 0.055(180 / 360)}{1 + 0.059(120 / 360)} = 1.0077.$$

The exact value is 1.00768225, a more precise number we shall need later.

Note that even though the discount rate exceeds the coupon, this bond is not selling at a discount, because it will pay $(1 + .055(180/360))$ in 120 days. The magnitude of the final payments exceeds the effect of the higher discount rate applied for the final 120 days. Of course, a sufficiently high discount rate would make the security sell at a discount.

Now let us take a look at the sensitivity of the floating-rate security to changes in interest rates.

The Interest Sensitivity of a Floating-Rate Security

Interest sensitivity is typically captured with the concepts of duration and convexity, which are obtained from derivatives of the security price with respect to the discount rate.

The Duration of a Floating-Rate Security

In general, for a bond with value V and yield r , duration is defined as

$$D = -\left(\frac{\partial V}{\partial r}\right)\left(\frac{1}{V}\right)(1 + r).$$

Oftentimes, duration is used in the form of modified duration, $D/(1 + r)$. Hence,

$$D^m = -\left(\frac{\partial V}{\partial r}\right)\left(\frac{1}{V}\right),$$

where $D^m = D/(1 + r)$. Duration is a measure of the sensitivity of a bond price to an interest rate. This notion is usually seen with the following formula, derived from the equation directly above:

$$\frac{\partial V}{V} = -D^m \partial r.$$

Of course, in the form above, changes in the bond price or interest rate are in the form of calculus derivatives. The above equation is usually approximated with the discrete relationship

$$\frac{\Delta V}{V} = -D^m \Delta r.$$

That is, we take the change in the market discount rate and multiply by the modified duration, changing the sign to reflect the inverse relationship between interest rates and bond prices.

Now let us consider our floating rate security priced at time t . Differentiating the price of the floating-rate security with respect to $r_{t,j+1}$, we obtain⁴

$$\begin{aligned} \frac{\partial V_t}{\partial r_{t,j+1}} &= \frac{-(1 + c_{j,j+1}q_{j,j+1})q_{t,j+1}}{(1 + r_{t,j+1}q_{t,j+1})^2} = -\left(\frac{1 + c_{j,j+1}q_{j,j+1}}{1 + r_{t,j+1}q_{t,j+1}}\right)\left(\frac{1}{1 + r_{t,j+1}q_{t,j+1}}\right)q_{t,j+1} \\ &= -V_t \left(\frac{1}{1 + r_{t,j+1}q_{t,j+1}}\right)q_{t,j+1}. \end{aligned}$$

From the definition of duration, we obtain the modified duration of a floating-rate security as

$$D_t^m = \frac{q_{t,j+1}}{1 + r_{t,j+1}q_{t,j+1}}.$$

Thus, this value can be used as a measure of the interest sensitivity of a floating-rate security.

Note that the duration of a floating-rate security is a slightly discounted measure of the remaining time until the coupon is reset. This formulation highlights how a floating-rate security, though possibly a long-term instrument, is effectively a short-term instrument, behaving like a zero coupon bond with maturity equal to the time remaining until the coupon is reset.⁵

A Numerical Example of the Duration of a Floating-Rate Security

In the example we used above, there are always 180 days between coupon dates; thus, $q_{j,j+1} = 180/360$. The duration on a payment date will be $0.5/(1 + r_{j,j+1}q_{j,j+1})$. Thus, on day 540 (using the figures from the example above),

$$D_{540}^m = \frac{0.5}{1 + 0.055(180 / 360)} = 0.4866.$$

⁴In taking this derivative, note that the coupon stays the same. The market discount rate is the variable of differentiation.

⁵In fact, the remaining time divided by the discount factor is precisely the duration of a zero coupon security.

On day 600, we need $q_{t,j+1}$, which is 120/360. Using the figures from the example above, we obtain duration of

$$D_{540}^m = \frac{(120 / 360)}{1 + 0.055(120 / 360)} = 0.3273.$$

The Convexity of a Floating-Rate Security

In practice, the interest sensitivity of a floating-rate security is relatively low because the coupons are usually reset no less often than once a year. This characteristic renders the security to be effectively a short-term security, which would give it relatively low interest-rate sensitivity. There is no theoretical reason why the period between coupon-reset dates need be short, however, so there could be floating-rate securities with much greater interest sensitivity. In that case, duration might be an inadequate measure of the interest sensitivity. Then we usually supplement duration with convexity.

Convexity, a second-order measure of interest sensitivity can be derived by taking the second derivative. Convexity of any bond is defined as

$$C = \left(\frac{\partial^2 V}{\partial r^2} \right) \left(\frac{1}{V} \right) (1 + r)^2.$$

Modified convexity is

$$C^m = \left(\frac{\partial^2 V}{\partial r^2} \right) \left(\frac{1}{V} \right),$$

with $C^m = C/(1 + r)^2$. For a floating-rate security, convexity is found as follows:

$$\begin{aligned} C &= \left(\frac{\partial^2 V}{\partial^2 r_{t,j+1}} \right) (1 + r_{t,j+1})^2 \left(\frac{1}{V} \right) \\ &= \left(\frac{2Vq_{t,j+1}^2}{(1 + r_{t,j+1})^2} \right) (1 + r_{t,j+1})^2 \left(\frac{1}{V} \right) \\ &= 2q_{t,j+1}^2. \\ C^m &= \frac{2q_{t,j+1}^2}{(1 + r_{t,j+1})^2} \end{aligned}$$

This expression is much simpler than the convexity formula for ordinary coupon bonds.

The convexity term can be used to improve the accuracy of the formula for the relationship between the change in the discount rate and the change in the price in the following manner:

$$\frac{\Delta V}{V} = -D^m \Delta r_{t,j+1} + \frac{1}{2} C^m (\Delta r_{t,j+1})^2.$$

This expression comes from a Taylor series expansion of the change in V , with differences (Δ 's) substituted for differentials to reflect the discrete time approximation.

Estimating the Interest Sensitivity with Duration and Convexity

Let us now move to day 600 with the upcoming coupon set at 5.5% and the discount rate at 5.9% where we found that the value is 1.0077. Now inject an immediate yield change of 100 basis points to 6.9%. Then the value would be

$$V = \frac{1 + 0.055(180 / 360)}{1 + 0.069(120 / 360)} = 1.0044.$$

The exact value is 1.00439883, which is a percentage change of $1.00439883/1.00768225 - 1 = -0.003258$.

The modified duration is 0.3273, and the modified convexity is $2(120/360)^2/(1 + 0.059(180/360))^2 = 0.2097$. The percentage change in the bond price should be

$$\begin{aligned} \frac{\Delta V}{V} &= -D^m \Delta r_{t,j+1} + \frac{1}{2} C^m (\Delta r_{t,j+1})^2 \\ &= -0.3273(0.01) + \frac{1}{2} (0.2097)(0.01)^2 \\ &= -0.003273 + 0.00001 = -0.003272. \end{aligned}$$

This figure is very close to the actual percentage change, but in this case, the duration by itself is a pretty good measure of the interest sensitivity, as it showed a percentage change of -0.003273. Convexity added very little, but it did move the estimate a little closer to the true change of -0.003258. As noted, the traditionally short period between coupon dates makes a floating-rate security not particularly sensitive to interest rate changes. Duration will oftentimes be sufficient as a measure of interest-rate sensitivity for short-maturity instruments. If we were dealing with floating-rate instruments in which there is a long time between the coupon changes, the interest sensitivity would likely be greater, and convexity would probably be more useful.

Concluding Comments

Slight adjustments can easily be made to derive the price, duration, and convexity under other forms of compounding-discounting. The results are qualitatively the same.

There are some more exotic forms of floating-rate securities. Some specify that the coupon is a multiple of the rate. These are called *leveraged floaters*. Others have the coupon move inversely, sometimes at a multiple, of the market rate. These are called *inverse floaters*. Some instruments have limits on how high or how low the coupons can go. All of these instruments are more complex than a conventional floating-rate security but can be analyzed in a similar manner to what we have done here.

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