

**TN05-02:
CALCULATING THE GREEKS IN THE BINOMIAL MODEL**

Version date: August 21, 2008

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This teaching note illustrates how to calculate the option Greeks using the binomial model. First let us set up the model. Let S be the current stock price and u and d be the up and down factors. The discrete risk-free rate per period is r . X is the exercise price of the option. We illustrate the standard binomial model with the following diagram.

Time 0	Time 1	Time 2
		S^{++} C^{++}
	S^+ C^+	
S C		S^{+-} C^{+-}
	S^- C^-	
		S^{--} C^{--}

We make the assumption that the down factor is the inverse of the up factor. This need not be the case but does simplify things a little.

Standard Approach

Delta

The option delta is defined as

$$\Delta = \frac{\partial C}{\partial S}.$$

In discrete time, we must approximate delta with the following:

$$\Delta = \frac{\Delta C}{\Delta S},$$

where the delta symbols in the numerator and denominator reflect the change in the prices of the call and stock from their current prices. The problem in measuring delta in a binomial tree is that technically we must measure these changes without changing time. From the binomial tree diagram, we see that the stock price can change only over time, not at a given instant in time. One way to deal with this problem is to have enough binomial periods so that the length of each period is very short. There is another way to fit the tree to deal with the problem, which we explore in the second half of this note. For now let us measure delta the best we can, given the above form of the tree.

In a continuous time world, the option price is non-linear with respect to the stock price, meaning that the option price change is not the same upward as it is downward. This is, of course, the convexity of the option price-stock price relationship. That convexity is evident in the binomial model is well. Looking at the tree, note that the call price can change from C to C^+ as the stock price changes from S to S^+ or from C to C^- as the stock price changes from S to S^- . Although it is not apparent in the above diagram, the option price change is not the same upward as it is downward. In continuous time, we deal with this problem by using the derivative, which is the limit of the rate of change in C for an infinitesimal change in S . The derivative linearizes the change in the option price in relation to the change in the stock price. In the binomial model, the usual approach to measuring delta is to take the average of the two possible call price changes and divide by the average of the two possible stock price changes:

$$\begin{aligned}\Delta &= \frac{(1/2)(C^+ - C) + (1/2)(C - C^-)}{(1/2)(S^+ - S) + (1/2)(S^- - S)} \\ &= \frac{C^+ - C^-}{S^+ - S^-}.\end{aligned}$$

A simple derivation of the binomial model shows that this is the correct measure of delta, because it is the number of shares that should be held to hedge in relation to a short position in one call option to form a perfect hedge. This is the same notion of delta in the continuous time world of the Black-Scholes model.

Gamma

A short option position hedged with delta shares of a long stock position will be a perfect hedge provided that the effect of the stock price change is accurately captured by the delta. This will occur only if the stock price change is infinitesimal. Hence, delta-hedged positions are at least partially vulnerable to virtually any change in the stock price. In addition, the delta will change with any change in time. If the delta changes, the hedged position must be revised to reflect a new number of shares. If the delta is not revised fast enough, the hedged position will lose its effectiveness. The gamma reflects the change in the delta given a change in the stock price and captures this effect.¹

The gamma is defined as

$$\Gamma = \frac{\partial \left(\frac{\partial C}{\partial S} \right)}{\partial S} = \frac{\partial^2 C}{\partial S^2}.$$

In the binomial model, we measure gamma by first noting that delta can change from the above measure, Δ , to either the new delta if the stock goes up, which we call Δ^+ , or to the new delta if the stock goes down, which we call Δ^- . Using the formula for delta, these new deltas are defined as follows:

$$\Delta^+ = \frac{C^{++} - C^{+-}}{S^{++} - S^{+-}}$$

$$\Delta^- = \frac{C^{+-} - C^{--}}{S^{+-} - S^{--}}.$$

In the binomial model, the gamma is estimated by averaging the change in the delta, $(1/2)(\Delta^+ - \Delta) + (1/2)(\Delta - \Delta^-) = (1/2)(\Delta^+ - \Delta^-)$, and dividing by the average change in the stock price, $(1/2)(S^{++} - S) + (1/2)(S - S^{--})$. This gives us

$$\Gamma = \frac{\left(\frac{C^{++} - C^{+-}}{S^{++} - S^{+-}} \right) - \left(\frac{C^{+-} - C^{--}}{S^{+-} - S^{--}} \right)}{(1/2)(S^{++} - S^{--})}.$$

Note that in order to calculate the gamma, you need the stock and option prices two time periods ahead. Thus, one important limitation of this approach is that it is impossible to calculate the gamma one time step before expiration. This point in time,

¹Technically gamma is somewhat meaningless in a binomial world, unless we impose some limitation on our ability to trade to adjust the hedge position whenever the stock price changes.

very close to expiration, is typically when the gamma is the highest and, therefore, most critical.

Theta

The theta is defined as the effect of the change in time on the option price. The formal specification for a continuous time world is

$$\Theta = \frac{\partial C}{\partial t},$$

where t is simply the point in time. If the remaining time is $\tau = T - t$, then $\partial t = T - \partial \tau$. Sometimes theta is evaluated in terms of $\partial \tau$, but we do not do so here as the transformation is simple.

Observing the binomial tree diagram, it should be apparent that an estimate of theta is obtained as

$$\Theta = \frac{C^{+-} - C}{2\Delta t},$$

where Δt is the length of each time step. Note that we are measuring the change in option price without the effect of a stock price change. Two time steps are required for the stock to return to where it started.²

Vega and Rho

Vega and rho are the effects of the volatility and interest rate, respectively, on the option price. Vega and rho cannot be measured as easily as delta, gamma, and theta. To understand why, it is necessary to understand the nature of volatility and the interest rate in a typical option pricing models.

The volatility and interest rate are assumed constant over the life of the option. When working in a continuous time world, we can certainly take the derivative of the Black-Scholes model with respect to the volatility or interest rate, but their interpretation is simply that the option price will be different if we use a different volatility or interest rate. But within the life of the option, the volatility and interest rate are assumed constant. This point should be very clear in the binomial model. The tree changes only as a result of time and the stock price change. The volatility and interest rate affect the

²Here we note that if the down factor is not the inverse of the up factor, then $S_{ud} \neq S$, and this method will not be as precise. Nonetheless, in most cases S_{ud} will be close enough to S to ignore any such problems.

up and down factors and the risk neutral probability, but these parameters are constant throughout the tree.

If we want to measure the sensitivity of the binomial option price to a different volatility or interest rate, we would simply recalculate the option value with a different volatility or interest rate, keeping in mind that the sensitivity is different depending on whether we use a higher or lower volatility or interest rate. We would try both an increase and a decrease in the volatility and interest rate. We would then average the option price difference and divide by the average difference of the volatility or interest rate to get a reasonable estimate of the vega or rho.³

Finally, let us note that the method described here for estimating vega and rho can also be used to estimate delta and gamma. Pelsser and Vorst (1994), however, report that the results are not very good in comparison to the method we discuss in the next section.

A More Precise Method for Estimating Delta and Gamma

Because delta and gamma should be measured without a change in time, a modified method of fitting the binomial tree can be used to achieve this property. Suppose we wish to fit a binomial tree with N time steps. We fit the tree with $N + 2$ time steps, and position the current stock price in the middle of the second time step, in the following manner. In our example in the previous section, we let $N = 2$. Then the tree would look as follows:

³We should also make sure that the change in volatility and interest rate is very small, given that we are attempting to approximate a derivative.

Time -1	Time 0	Time 1	Time 2	Time 3
				S ⁺⁺⁺⁺ C ⁺⁺⁺⁺
			S ⁺⁺⁺ C ⁺⁺⁺	
		S ⁺⁺ C ⁺⁺		S ⁺⁺⁺⁻ C ⁺⁺⁺⁻
	S ⁺ C ⁺		S ⁺⁺⁻ C ⁺⁺⁻	
S C		S ⁺⁻ C ⁺⁻		S ⁺⁺⁻⁻ C ⁺⁺⁻⁻
	S ⁻ C ⁻		S ⁺⁻⁻ C ⁺⁻⁻	
		S ⁻⁻ C ⁻⁻		S ⁺⁻⁻⁻ C ⁺⁻⁻⁻
			S ⁻⁻⁻ C ⁻⁻⁻	
				S ⁻⁻⁻⁻ C ⁻⁻⁻⁻

The current stock price is S^+ , and the current option price is C^+ . Note that if $d = 1/u$, S^+ will be the same as S , although C^+ will not be the same as C .

This binomial tree offers several advantages over the standard approach. As we shall see in the next sections, it will allow us to calculate both delta and gamma without imposing a time change.⁴

Delta

Positioned at the current stock price of S^+ , we can now measure the effect of a different stock price on the option price without moving forward in time. We can

⁴With this type of tree, if we need to measure gamma one period prior to expiration, we can do so.

observe the effect of the stock price being S^{++} or S^{--} at the same time point and measure delta as the average option price change divided by the average stock price change.

$$\begin{aligned}\Delta &= \frac{(1/2)(C^{++} - C^{+-}) + (1/2)(C^{+-} - C^{--})}{(1/2)(S^{++} - S^{+-}) + (1/2)(S^{+-} - S^{--})} \\ &= \frac{C^{++} - C^{--}}{S^{++} - S^{--}}.\end{aligned}$$

Gamma

Remember that gamma measures the change in delta, given a change in the stock price. From the point at which the stock is at S^{+-} , we can think of the gamma as being the difference in the change in option price over the change in stock price for the stock price in the node above and for the stock price in the node below. That is,

$$\begin{aligned}\Gamma &= \frac{\left(\frac{C^{++} - C^{+-}}{S^{++} - S^{+-}}\right) - \left(\frac{C^{+-} - C^{--}}{S^{+-} - S^{--}}\right)}{(1/2)(S^{++} - S^{+-}) + (1/2)(S^{+-} - S^{--})} \\ &= \frac{\left(\frac{C^{++} - C^{+-}}{S^{++} - S^{+-}}\right) - \left(\frac{C^{+-} - C^{--}}{S^{+-} - S^{--}}\right)}{(1/2)(S^{++} - S^{--})}.\end{aligned}$$

In other words, we measure the gamma as the difference in the delta with S at the higher price and delta with S at the lower price, divided by the difference in the stock prices. Note that we can now measure gamma at any time step, including the one before expiration.

Theta

As in the standard case, theta will require two time steps ahead of the current price. The theta would then be measured the same as in the standard case:

$$\Theta = \frac{C^{++-} - C^{+-}}{2\Delta t}.$$

Vega and Rho

The same issues concerning the estimation of vega and rho would apply here. The tree could be recomputed with different volatilities and interest rates to obtain estimates of the vega and rho.

Numerical Examples

Now let us take a look at some numerical examples. To be able to observe the tree, we first look at a simple problem with two binomial time steps. Later we extend the problem to a large number of time steps. We use the following inputs: $S = 100$, $X = 100$, $T = 2$, $\sigma = 0.4$, $r = 0.07$ (discrete), and $n = 2$. Using the common parameterization of

$$u = e^{\sigma\sqrt{T/n}}$$

$$d = 1/u$$

$$r(\text{per period}) = (1 + r)^{T/n},$$

we obtain $u = 1.4918$, $d = 0.6703$, and the per period risk-free rate is 0.07. The length of a time step is $2 \text{ (years)}/2 = 1$. The standard tree is as follows:⁵

Time 0	Time 1	Time 2
		$S^{++} = 222.55$ $C^{++} = 122.55$
	$S^+ = 149.18$ $C^+ = 55.72$	
$S = 100$ $C = 25.34$		$S^{+-} = 100$ $C^{+-} = 0$
	$S^- = 67.03$ $C^- = 0$	
		$S^{--} = 44.93$ $C^{--} = 0$

The Greeks are computed as follows:

$$\Delta = \frac{C^+ - C^-}{S^+ - S^-} = \frac{55.72 - 0}{149.18 - 67.03} = 0.6783$$

$$\Gamma = \frac{\left(\frac{C^{++} - C^{+-}}{S^{++} - S^{+-}}\right) - \left(\frac{C^{+-} - C^{--}}{S^{+-} - S^{--}}\right)}{(1/2)(S^+ - S^-)} = \frac{\left(\frac{122.55 - 0}{222.55 - 100}\right) - \left(\frac{0 - 0}{100 - 44.93}\right)}{(1/2)(222.55 - 44.93)} = 0.0113$$

$$\Theta = \frac{C^{+-} - C}{2\Delta t} = \frac{0 - 25.34}{2(1)} = -12.66.$$

⁵We do not explain how the option price is obtained, as this is widely covered elsewhere and is considered prerequisite knowledge to reading this Teaching Note.

The tree for the improved method

Time -2	Time -1	Time 0	Time 1	Time 2
				$S^{++++} = 495.30$ $C^{++++} = 395.30$
			$S^{+++} = 332.01$ $C^{+++} = 238.55$	
		$S^{++} = 222.55$ $C^{++} = 135.21$		$S^{+++} = 222.55$ $C^{+++} = 122.55$
	$S^+ = 149.18$ $C^+ = 73.64$		$S^{++} = 149.18$ $C^{++} = 55.72$	
$S = 100$ $C = 39.01$		$S^{+-} = 100$ $C^{+-} = 25.34$		$S^{++} = 100$ $C^{++} = 0$
	$S^- = 67.03$ $C^- = 11.52$		$S^{+-} = 67.03$ $C^{+-} = 0$	
		$S^{--} = 44.93$ $C^{--} = 0$		$S^{+++} = 44.93$ $C^{+++} = 0$
			$S^{--} = 30.12$ $C^{--} = 0$	
				$S^{----} = 20.19$ $C^{----} = 0$

The current state is the middle position of time 0 where the stock is at 100 and the option is at 25.34.⁶ Delta and Gamma are computed as follows:

$$\Delta = \frac{C^{++} - C^{--}}{S^{++} - S^{--}} = \frac{135.21 - 0}{222.55 - 44.93} = 0.7612$$

$$\Gamma = \frac{\left(\frac{C^{++} - C^{+-}}{S^{++} - S^{+-}}\right) - \left(\frac{C^{+-} - C^{--}}{S^{+-} - S^{--}}\right)}{(1/2)(S^{++} - S^{--})} = \frac{\left(\frac{135.21 - 25.34}{222.55 - 100}\right) - \left(\frac{25.34 - 0}{100 - 44.93}\right)}{(1/2)(222.55 - 44.93)} = 0.0049.$$

⁶Note that this is the same price as in the standard method.

Theta is the same as in the standard method, but this will not always be the case. The period two time steps ahead in both models is the expiration, so the option value will be the same. But for models with more time steps, the thetas will not always be the same.

The improved method probably works better for limited number of time steps, but with the fast computers of today, most binomial computations can be done with a large number of time steps. In that case, the length of each time step would be very short and the standard method will give pretty accurate measures of delta and gamma. Let us work a problem with a large number of time steps. Of course, these answers should converge to those obtained from the Black-Scholes model. Working the same problem with 500 time steps, we get the following results, which we show alongside the Black-Scholes values:

	Standard Binomial	Improved Binomial	Black-Scholes
C	27.7556	27.7556	27.7663
Δ	0.6991	0.6995	0.6992
Γ	0.0062	0.0061	0.0062
Θ	-7.7853	-7.7781	-7.7751

Again, for both methods the option prices are the same. Note that the standard binomial method gives slightly closer values to the Black-Scholes delta and gamma but the differences are barely detectable. The improved binomial mode gives a slightly closer estimate of theta. All in all, however, with a sufficient number of time steps there is no need to use the improved method, because the length of each time step is extremely small in the standard method.

References

Some material on binomial computation of the Greeks is in:

Chriss, Neil A. *Black-Scholes and Beyond: Option Pricing Models*. Chicago: Irwin Professional Publishing (1997).

Chung, San-Lin and Mark Shackleton. "The Binomial Black-Scholes Model and the Greeks." *The Journal of Futures Markets* 22 (2002), 143-153.

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