

**TN05-03:  
A GENERALIZATION  
OF THE COST OF CARRY FORWARD/FUTURES PRICING MODEL**

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The purpose of this note is to develop a generalized formulation of the cost of carry model for pricing forward and futures contracts.<sup>1</sup> The model is widely known but appears in so many different forms that it seems as if there is more than one model. In fact, there is but one model with many variations adapted to fit different types of underlying assets and special conventions found in the respective markets. The four primary types of underlying assets are stocks, currencies, commodities, and fixed-income instruments, sometimes called bonds, notes, or bills. For certain types of assets, there are different ways of modeling the cash flows on the asset and different ways of designing such contracts.

We start by setting up some initial concepts. First, we shall follow the practice that an amount of dollars or any other currency will be denoted with capital letters, while rates will be denoted with lower case letters. In some cases, the letters are Roman, and in some they are Greek. Subscripts will indicate a value observed at the time stated by the subscript. Arguments in parentheses provide other necessary information.

**Time and the Time Value of Money**

We start by specifying that time runs from  $t = 0, \dots, T$  where the contract is initiated at  $t = 0$  and expires at  $t = T$ . Without loss of generality, we can let  $t$  be a simple day count. Hence  $T$  is the number of days from the start of the contract until expiration.

Now let us establish a measure of the time value of money. As simple as this concept seems to be, it is a source of much confusion in pricing forwards/futures and accounts for a great deal of the variation observed across pricing formulas.

Let  $B_t(T)$  be the value at time  $t$  of \$1, or any currency considered to be the home currency of the person trading the contract, at time  $T$ . More often than not,  $B_t(T)$  is expressed in terms of an interest rate. But it is extremely important to understand the following:

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<sup>1</sup>In most cases, we shall refer to these contracts as forward/futures contracts. Typically futures contracts are priced like forward contracts, but there are differences, which we do not take up here.

$B_t(T)$  is a present value: it expresses the relationship between \$1 for certain in the future and its value today.

It is the price of a claim on \$1 at a future date. The value  $B_t(T)$  is determined by supply and demand in the short-term risk-free interest rate market.<sup>2</sup> An interest rate, on the other hand, is simply a function that relates present to future value. That is,

*An interest rate is nothing more than a mathematical relationship that maps a present value to a future value or vice versa. There are an infinite number interest rates specifications that can convert the value  $B_t(T)$  to \$1 at T, or vice versa.*

Now let us look at four common interest rate specifications.

#### *Discrete Interest with Annual Compounding*

With discrete interest with annual compounding, interest is accumulated only at the end of a year. Let  $r_t(T)$  be the interest rate observed at time t for discounting \$1 at time T in the following manner:

$$B_t(T) = \frac{1}{(1 + r_t(T))^{(T-t)/365}}. \quad (1)$$

Here  $T - t$  is the number of days over the period and  $(T - t)/365$  is the number of years over the period.

It is, of course, possible to compound more frequently than once a year. This convention is common in interest-bearing bank deposits, which often compound quarterly, monthly, daily, or continuously. Using monthly as an example,  $r_t(T)$  would be re-stated as a monthly rate and  $(T-t)/365$  would be the number of months in the compounding period.

#### *LIBOR (London Interbank Offer Rate or Eurodollar) Interest*

The Eurodollar market always assumes 360 days in a year and computes interest using a method known as add-on interest. Let  $i_t(T)$  (the Greek letter iota) be the LIBOR rate on day t for discounting \$1 on day T. \$1 invested for T days at the rate  $i_t(T)$  would grow to a value of  $1 + i_t(T)[(T-t)/360]$ .<sup>3</sup> Hence, \$1 at T has a present value of:

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<sup>2</sup>In more fundamental terms,  $B_t(T)$  describes the concept of “time preference,” a notion that reflects the cost of waiting until a later date to receive money. We assume no inflation or risk, so  $B_t(T)$  simply captures the discount a person applies to a known future value to obtain its present value.

<sup>3</sup>Note that the “(T)” indicates that the rate applies to the period ending at T. The bracketed term  $[(T-t)/360]$  means that the rate is multiplied by  $(T-t)/360$ .

$$B_t(T) = \frac{1}{1 + r_t(T)[(T-t)/360]}. \quad (2)$$

### *Discount Interest*

The U. S. Treasury Bill market and a few other markets use discount interest, a method in which interest is deducted from the principal in advance of the loan, thereby raising the effective rate. Letting  $r_t^d(T)$  be the discount rate, the interest rate formulation is as follows:

$$B_t(T) = 1 - r_t^d(T)[(T-t)/360]. \quad (3)$$

Note that this specification results in  $r_t^d(T)[T/360]$  deducted from the principal of \$1, giving the present value  $B_t(T)$ .

### *Continuous Interest*

Continuously compounded interest is commonly used in many financial models and greatly simplifies some of the mathematics. Let  $r_t^c(T)$  be the continuously compounded rate appropriate for discounting on day  $t$  of \$1 to be paid on day  $T$ . When interest compounds continuously, \$1 grows to a value of  $\exp(r_t^c(T)[(T-t)/365])$ . Hence, \$1 at  $T$  has a present value of

$$B_t(T) = e^{-r_t^c(T)[(T-t)/365]}. \quad (4)$$

### *Final Comments on Interest Specifications*

Recall that in all of these cases,  $B_t(T)$  is the same: the value on day  $t$  of \$1 to be paid for sure on day  $T$ . Regardless of how interest is specified, the value assigned to a specific future payoff must be the same. One can quote interest any of the above ways or even other ways not covered here. But the value of a sure receipt of \$1 at a future date must be the same regardless of how interest is quoted.

Before going further, you should firmly establish the principle that to find the present value as of  $t$  of a future amount at  $T$ , multiply that amount by  $B_t(T)$ . To compound a current amount from  $t$  to  $T$ , divide it by  $B_t(T)$ .

### **Specifying the Terminology for Forward/Futures Pricing**

Let  $S_t$  be the price of the underlying spot market asset at time  $t$  ( $t = 0, \dots, T$ ) and  $F(0,T)$  be the forward/futures price for a contract established at time 0 and expiring at time  $T$ . We now take a look at some factors that affect forward/futures prices and designate symbols for each.

### *Cash Flows on the Underlying Asset*

If the underlying asset generates any positive cash flows, such as dividends on a stock or coupon interest on bonds, during the life of the contract (the period  $t$  to  $T$ ), those cash flows accrue to a value of  $\Lambda_T(t,T)$ . We can think of  $\Lambda_T(t,T)$  as the cash flows paid on the asset during the period  $t$  to  $T$  under the assumption that they are collected and reinvested at the risk-free rate.

### *Storage Costs on the Underlying Asset*

If the asset incurs any costs from storage over the period  $t$  to  $T$ , those costs accrue to a value of  $\Theta_T(t,T)$ . We can think of  $\Theta_T(t,T)$  as the costs incurred from holding the asset during the period  $t$  to  $T$  under the assumption that the party supplying the storage service bills the holder of the asset at the end of the storage period.<sup>4</sup>

### *Non-Pecuniary Benefits on the Underlying Asset*

Some commodities are considered to offer a non-pecuniary return, sometimes called a convenience yield. Any such benefits accrue to a value of  $\Gamma_T(t,T)$ . Being non-pecuniary benefits, it is difficult to conceptualize the notion that these benefits are collected and reinvested. Indeed the entire notion of non-pecuniary benefits is a somewhat fuzzy concept, not easily explicable or measurable. Nonetheless, economists usually agree that such benefits can exist and play a role in pricing forward/futures contracts. Just think of  $\Gamma_T(t,T)$  as the value at  $T$  of these non-pecuniary benefits earned on the asset during the period  $t$  to  $T$ .

### *Costs and Benefits Expressed as Present Values*

These three concepts, all attributable to ownership of the asset over the period  $t$  to  $T$ , can also be expressed in present value terms:

$$\Lambda_t(t,T) = \Lambda_T(t,T)B_t(T)$$

$$\Theta_t(t,T) = \Theta_T(t,T)B_t(T)$$

$$\Gamma_t(t,T) = \Gamma_T(t,T)B_t(T),$$

which will be more convenient on occasion. Naturally, because these values are known at  $T$ , they are known at  $t$ .

### **The Standard Cost of Carry Model**

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<sup>4</sup>Alternatively we can think of the party holding the asset as borrowing to pay the storage costs and paying off the loan when the contract expires.

Consider the following strategy: purchase the generic asset at time 0 at price  $S_0$  and enter into a forward/futures contract to deliver it at time  $T$  for the price  $F(0,T)$ . At time  $T$ , accrued benefits will equal  $\Lambda_T(0,T)$  and  $\Gamma_T(0,T)$  and accrued costs will equal  $\Theta_T(0,T)$ . At  $T$ , deliver the asset and receive  $F(0,T)$ . Total value at  $T$  is  $F(0,T) + \Lambda_T(0,T) + \Gamma_T(0,T) - \Theta_T(0,T)$ . This expression contains only terms known at  $t = 0$ . Hence, the transaction is risk-free when initiated, and its present value must be the amount of the initial investment,  $S_0$ . This equivalence is expressed as

$$B_0(T)(F(0,T) + \Lambda_T(0,T) + \Gamma_T(0,T) - \Theta_T(0,T)) = S_0$$

Solving for the forward/futures price, we obtain

*Cost of Carry Model – Future Value Formulation:*

$$F(0,T) = \frac{S_0}{B_0(T)} + \Theta_T(0,T) - \Lambda_T(0,T) - \Gamma_T(0,T). \quad (5)$$

The forward/futures price is the spot price compounded at the risk-free rate plus the accrued (future) value of any costs minus the future value of any benefits. An alternative way of writing this expression is:

*Cost of Carry Model – Present Value Formulation:*

$$F(0,T) = \frac{S_0 + \Theta_0(0,T) - \Lambda_0(0,T) - \Gamma_0(0,T)}{B_0(T)}, \quad (6)$$

where we take the spot price, add the present value of costs, subtract the present value of benefits, and compound the result at the risk-free rate.

These formulas must accurately reflect the forward/futures price through the forces of arbitrage. For example, if the forward price in the market exceeds that given in Equations (5) or (6), arbitrageurs will sell the forward/futures contract, buy the underlying, and hold the position to expiration. The return, after accounting for any benefits and costs, will exceed the risk-free return and have no risk. Selling pressure will then drive the forward/futures price down. If it falls below the price given in Equations (5) or (6), arbitrageurs will buy the forward/futures contract, sell short the underlying, and hold the position to expiration, resulting in a short position that implicitly borrows at a rate better than the risk-free rate. Buying pressure will then drive the forward/futures price up. It must then stop at the value given by Equations (5) or (6). Of course, transaction costs could discourage arbitrage if the deviation of the market price from

Equations (5) or (6) is not sufficiently large. In addition, short sale restrictions, as is common in commodity markets, can result in market prices lower than those given in Equations (5) or (6). For this note, we shall assume that such frictions do not exist.

This model is generic and holds for any assets that can be stored and delivered or in which the contract cash-settles at expiration to the price of the underlying asset.<sup>5</sup> Characteristics of certain assets enable us to specify the equation more precisely and lead to formulas that seem quite different from these, though as we shall show, all of the formulas come back to Equations (5) and (6).

We shall now look at the cost of carry model for stocks, bonds, currencies, and commodities as the underlying asset.

#### *The Cost of Carry Formulation for Stocks*

For stocks there are no costs of storage or non-pecuniary benefits. Hence,  $\Theta_T(t,T)$  and  $\Gamma_T(t,T)$  and their current values are zero. Naturally dividends are commonly paid on stocks, usually on a quarterly basis in the U. S. Suppose over the life of the contract, there are  $i = 1, \dots, n$  dividends in the amounts  $D_1, D_2, \dots, D_n$ , which occur at times  $t_1, t_2, \dots, t_n$  where  $0 \leq t_1$  and  $t_n \leq T$ . The present and future values of the dividends are found as

$$\Lambda_0(0,T) = \sum_{i=1}^n B_0(t_i)D_{t_i}$$

$$\Lambda_T(0,T) = \frac{\Lambda_0(0,T)}{B_0(T)} = \frac{\sum_{i=1}^n B_0(t_i)D_{t_i}}{B_0(T)} = \sum_{i=1}^n \frac{D_{t_i}}{B_{t_i}(T)}.$$

Using these formulas, one easily obtains the forward/futures price with Equations (5) or (6).

A fairly common practice is to use a continuous yield formulation for the dividends. This approach assumes that dividends are paid continuously at a known rate. This approach is often used with index options in which the index includes many stocks, which pay dividends at somewhat diverse times. Here the stock price minus the present

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<sup>5</sup>The highly successful Eurodollar futures contract of the Chicago Mercantile Exchange and the moderately successful Federal Funds futures contract of the Chicago Board of Trade do not have precise convergence at expiration. Hence, they cannot technically be priced by these formulas.

value of the dividends is  $S_0 e^{-\delta_0^c(T)[T/365]}$  where  $\delta_0^c(T)$  is the continuous dividend yield.

Therefore, we must have

$$S_0 - \Lambda_0(0, T) = S_0 e^{-\delta_0^c(T)[T/365]}.$$

This means that

$$\Lambda_0(0, T) = S_0 \left(1 - e^{-\delta_0^c(T)[T/365]}\right).$$

With  $\Gamma_T(0, T) = 0$  and  $\Theta_T(0, T) = 0$ , we substitute  $S_0 e^{-\delta_0^c(T)[T/365]}$  for  $S_0 - \Lambda_0(0, T)$  in Equation (6) and obtain

$$F(0, T) = \frac{S_0 e^{-\delta_0^c(T)[T/365]}}{B_0(T)}.$$

When assuming a continuous dividend yield, we would tend to use the continuous specification of interest (Equation (4)), so the forward/futures price formula would be a very simple one:

$$F(0, T) = S_0 e^{(r_0^c(T) - \delta_0^c(T))[T/365]}.$$

### *The Cost of Carry Formulation for Currencies*

First we note that a foreign currency incurs no storage costs nor does it provide any non-pecuniary benefits. Hence,  $\Theta_T(0, T)$  and  $\Gamma_T(0, T)$  are both zero. Now note that an investment in a foreign currency is considered to be an investment in an interest-earning asset, because the currency pays interest at the foreign rate. Thus, the foreign interest can be viewed like a dividend on a stock. Hence, we can price currency forwards/futures in a manner similar to that of forwards/futures on stock. We let  $S_0$  be the current exchange rate, which represents units of domestic currency required to buy one unit of the foreign currency.

There is, however, one complicating factor. Recall that the value  $\Lambda_0(0, T)$  or  $\Lambda_T(0, T)$  is a known amount at time  $t = 0$ . Currencies accrue interest at the foreign interest rate. Hence, at time  $T$  the holder of a unit of foreign currency will have not only the unit of foreign currency but also interest that has accrued. While the amount of the interest will be known in units of foreign currency, its value in units of the party's domestic currency will be unknown because of exchange rate risk. We must deal with this problem by adapting the generalized cost of carry model to the case of a foreign currency.

We can do this by adjusting the forward/futures contract so that instead of covering one unit of the currency, it covers more than one unit, where the incremental units will be the interest earned on an investment of one unit over the period 0 to T. Let the discretely compounded foreign interest rate at time 0 be  $\rho_0(T)$ . Interest accrues on one unit of the currency over the period of 0 to T by the factor  $(1 + \rho_0(T))^{T/365}$ . If we invest  $S_0$  units of the domestic currency and purchase one unit of the foreign currency, at the end of the period, we shall have  $(1 + \rho_0(T))^{T/365}$  units of the foreign currency. This is the precise number of units required for delivery, provided we set the contract up to cover this many units. So, at the start, let us specify that the contract is for  $(1 + \rho_0(T))^{T/365}$  units of the currency.

So at T we make delivery and receive  $F(0,T)(1 + \rho_0(T))^{T/365}$ . Hence, we invested  $S_0$  and received  $F(0,T)(1 + \rho_0(T))^{T/365}$ , a known amount. The payoff at T must have a present value equal to the amount invested at 0,  $S_0$ . Hence,  $F(0,T)(1 + \rho_0(T))^{T/365}B_0(T)$  must equal  $S_0$ . Using Equation (6), the forward/futures price is

$$F(0,T) = \frac{S_0(1 + \rho_0(T))^{-T/365}}{B_0(T)}.$$

We see that the spot rate is discounted at the foreign interest rate and compounded at the domestic interest rate.

In the formulation directly above,  $B_0(T)$  would most likely to take the form of Equation (1). In that case, the formula looks like

$$F(0,T) = S_0(1 + \rho_0(T))^{-T/365}(1 + r_0(T))^{T/365}.$$

It is somewhat common to see this formula written as the approximation,<sup>6</sup>

$$F(0,T) \approx S_0(1 + r_0(T) - \rho_0(T))^{T/365}.$$

Suppose the currency pays interest in the LIBOR manner. Let  $r_0^f(T)$  be the foreign LIBOR. Then, the forward/futures contract must be structured to require delivery of  $1 + r_0^f(T)[T/360]$  units of the currency. Then the forward/futures price will look slightly different at:

$$F(0,T) = \frac{S_0}{(1 + r_0^f(T)[T/360])B_0(T)}.$$

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<sup>6</sup>This result is shown to be approximately correct by taking the log of the expression  $(1 + \rho_0(T))^{-T/365}(1 + r_0(T))^{T/365}$  and using the result that  $\ln(1 + x) \approx x$  if  $x$  is small.

We see that the spot rate is discounted at the foreign rate and compounded at the domestic rate. Also, we should expect that if interest is assumed to be paid in the LIBOR manner, then it is appropriate to have  $B_0(T)$  follow the LIBOR formulation, Equation (2).

Now let us specify that interest accrues at the continuously compounded foreign rate,  $\rho^c_t(T)$ . Then the number of units of currency covered by the contract must be adjusted to  $e^{\rho^c_0(T)[T/365]}$ . Following the same steps, we obtain the forward/futures price as

$$F(0,T) = \frac{S_0 e^{-\rho^c_0(T)[T/365]}}{B_0(T)}.$$

The interest factor,  $B_0(T)$ , would probably be specified in the continuous manner as in Equation (4), giving

$$F(0,T) = S_0 e^{(r^c_0(T) - \rho^c_0(T))[T/365]}.$$

#### *The Cost of Carry Formulation for Commodities*

Commodities or physical assets such as oil and gold are always characterized by the presence of storage costs. Hence,  $\Theta_T(t,T)$  is positive. In addition, these assets provide a convenience yield. Thus,  $\Gamma_T(t,T)$  is positive. The cost of carry relationship for forwards/futures on commodities is well-described by Equations (5) and (6) with  $\Lambda_T(0,T) = 0$ , because there are typically no positive cash flows from commodities.

In some cases, the storage cost and convenience yields are described as rates. For example, suppose the storage costs accrue at a discrete rate of  $\theta_t(T)$ . Then an investment of  $S_0$  in the asset will accrue costs by the factor  $(1 + \theta_t(T))^{T/365}$ . Total storage costs at  $T$  will, thus, be  $\Theta_T(0,T) = S_0(1 + \theta_t(T))^{T/365} - S_0$ . Alternatively, let  $\gamma_t(T)$  be the convenience yield rate. Then, the convenience yield at  $T$  will be  $\Gamma_T(t,T) = S_0(1 + \gamma_t(T))^{T/365} - S_0$ . Substituting these results into Equation (5), we obtain

$$F(0,T) = \frac{S_0}{B_0(T)} + S_0((1 + \theta_t(T))^{T/365} - 1) - S_0((1 + \gamma_t(T))^{T/365} - 1).$$

If rates are expressed as continuous, we have storage costs of  $\Theta_T(0,T) = S_0(e^{\theta^c_0(T)[T/365]} - 1)$  and a convenience yield of  $\Gamma_T(0,T) = S_0(e^{\gamma^c_0(T)[T/365]} - 1)$ . Substituting into (5), we obtain

$$F(0,T) = \frac{S_0}{B_0(T)} + S_0(e^{\theta^c_0(T)[T/365]} - 1) - S_0(e^{\gamma^c_0(T)[T/365]} - 1).$$

As noted above,  $B_0(T)$  would probably also be in the continuous formulation. Then  $1/B_0(T)$  would be  $\exp[r_0^c(T)[T/365]$ . A good approximation that is often used is<sup>7</sup>

$$F(0,T) = S_0 e^{(r_0^c(T) + \theta_0^c(T) - \gamma_0^c(T))[T/365]}$$

Specification of commodity forwards/futures pricing with rates for storage costs and the convenience yield is a little more awkward than using dollar amounts.

#### *The Cost of Carry Formulation for Fixed-Income Securities*

Fixed-income securities can pose special challenges for the pricing of forward/futures contracts. One source of confusion arises from the quotation method used in the fixed-income market. In the days before calculators and computers simplified fixed-income calculations, it was found that bond prices could be determined most easily if the interest that had accrued from the last coupon payment date is separated from the full price and the remaining amount is the quoted price. Thus, a quoted price of 105 16/32, meaning \$105.50 per \$100 par, would not be the full price one would pay to purchase the bond. One would have to add the interest that has accrued since the last payment. This would be done by multiplying the interest payment by the fraction of the current interest payment period that has elapsed. For example, assuming that the next interest payment is \$4 (4% on \$100 par, 8% annual rate), 65 days have elapsed since the last interest payment, and there are 181 days between the last interest payment and the next, then the accrued interest is  $\$4(65/181) = \$1.44$ . The full price would be  $\$105.50 + \$1.44 = \$106.94$ . It is important to recognize that \$106.94 is the correct economic value of the bond and reflects the present value of all remaining coupons and the final principal, each discounted at the appropriate rate over the appropriate period. Bond-market convention, however, calls for deduction of the \$1.44 accrued interest so that the quoted price is \$105.50. Not surprisingly the bond futures market uses this same convention. Hence, a quoted forward/futures price does not reflect its full price, which includes the accrued interest. If we want our model to give us the quoted price, we shall need to adjust it.

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<sup>7</sup>This approximation works reasonably well because  $\exp(x) + \exp(y) + \exp(z) \approx \exp(x + y + z)$ . This result can be demonstrated using a second-order Taylor series expansion on each term and dropping expressions like  $xy$ ,  $xz$ , and  $yz$ , which are close to zero for reasonable orders of magnitude.

Another reason for the complexity of bond forward/futures pricing is that many forward/futures contracts do not specify a single deliverable bond and use a conversion factor system to attempt to render all bonds equally desirable for delivery. We will incorporate this point later, but for now, let us keep things as simple as possible.

First we note that bonds do not have storage costs or convenience yields. Hence,  $\Theta_T(t,T)$  and  $\Gamma_T(t,T)$  are both zero. Cash flows during the contract life may or may not exist. For zero coupon bonds, there are no such cash flows. Hence,  $\Lambda_T(t,T) = 0$ . Then the cost of carry forward/futures price is found from Equation (6) as

$$F(0,T) = \frac{S_0}{B_0(T)}.$$

Of course, this is simply the spot price compounded at the risk-free rate.

Now consider a forward/futures contract on a specific coupon-bearing bond.<sup>8</sup> We can easily adapt the result previously obtained for dividend-paying stocks to this case. Suppose over the life of the contract, there are  $i = 1, \dots, n$  coupon payments of  $C$ , which occur at times  $t_1, t_2, \dots, t_n$  where  $0 \leq t_1$  and  $t_n \leq T$ . The present and future values of the dividends are found as

$$\begin{aligned} \Lambda_0(0,T) &= \sum_{i=1}^n B_0(t_i)C = C \sum_{i=1}^n B_0(t_i) \\ \Lambda_T(0,T) &= \frac{\Lambda_0(0,T)}{B_0(T)} = \frac{C \sum_{i=1}^n B_0(t_i)}{B_0(T)} = C \sum_{i=1}^n \frac{1}{B_{t_i}(T)}. \end{aligned}$$

Using these formulas, one easily obtains the forward/futures price with Equations (5) or (6).

In Equations (5) and (6), the values  $S_0$  and  $F(0,T)$  represent the actual, correct prices, meaning that they include the accrued interest. Given the convention of quoting the forward/futures price without accrued interest, we usually adapt these equations to give us the quoted price, i.e., without accrued interest. Focusing on Equation (6), let us redefine the following relations:

$$S_0^Q = \text{quoted spot price}$$

$$AI_0 = \text{accrued interest as of time 0}$$

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<sup>8</sup>In other words, we mean that there is no possibility of delivering any bond other than the specific bond prescribed in the contract.

$F(0,T)^Q =$  quoted forward/futures price

$AI_T =$  accrued interest as of time T

Thus,

$$S_0 = S_0^Q + AI_0$$

$$F(0,T) = F(0,T)^Q + AI_T.$$

Then restating Equation (6) we obtain

$$F(0,T) = \frac{S_0 + \Theta_0(0,T) - \Lambda_0(0,T) - \Gamma_0(0,T)}{B_0(T)} \Rightarrow$$

$$F(0,T)^Q + AI_T = \frac{S_0^Q + AI_0 - \Lambda_0(0,T)}{B_0(T)} \Rightarrow$$

$$F(0,T)^Q = \frac{S_0^Q + AI_0}{B_0(T)} - \Lambda_T(0,T) - AI_T,$$

where you will note that we have adjusted the cash flows to the future date T.

The most complex case is when there is more than one deliverable bond. Here the forward/futures price is multiplied by a number called the conversion factor, which we shall denote as CF. The contract is designed as though the underlying bond has a specific coupon, called the benchmark coupon.<sup>9</sup> Any bond is eligible for delivery as long as its maturity exceeds a specific minimum.<sup>10</sup> The conversion factor is associated with each eligible bond deliverable on a particular contract. The CF for a bond delivered on one contract will differ from the CF for that same bond delivered on another contract. The CF for one bond on a contract will differ from the CF for a bond with a different coupon or maturity deliverable on that same contract. The CF is defined as the value of a \$1 par bond with coupon and maturity equal to those of the deliverable bond, priced to yield the benchmark rate. If a bond with a higher (lower) coupon than the benchmark rate is delivered, the CF exceeds (is less than) 1. In this manner, the party holding the short

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<sup>9</sup>One of the most actively traded contracts is the Chicago Board of Trade's U.S. Treasury Bond futures. It was originally designed with an 8% benchmark coupon but has since been adjusted to a 6% benchmark coupon.

<sup>10</sup>The minimum maturity or call data of eligible deliverable bonds for the CBOT bond contract is 15 years. CBOT note contracts have minimum and maximum maturities of 6 ½ to 10 years for the 10-year contract, four years, two months to five years, three months for the five-year contract, and one year, nine months to two years for the two-year contract.

position is compensated if he delivers a bond with a higher coupon than the benchmark and penalized if he delivers a bond with a lower coupon than the benchmark.<sup>11</sup>

Upon delivery, the buyer pays the seller the forward/futures price times the conversion factor plus the accrued interest as of delivery. Hence, we adapt Equation (6) as follows:

$$F(0,T)^{\varrho}(CF) + AI_T = \frac{S_0^{\varrho} + AI_0 - \Lambda_0(0,T)}{B_0(T)} \Rightarrow$$

$$F(0,T)^{\varrho} = \frac{\frac{S_0^{\varrho} + AI_0}{B_0(T)} - \Lambda_T(0,T) - AI_T}{CF}.$$

A special case is that the deliverable bond is the benchmark bond. In that case,  $CF = 1$  and we obtain the formula given directly above for a single deliverable bond.

### Concluding Comments

We have shown that all cost of carry models arise from a single formula. Characteristics unique to the specific spot and forward/futures markets, price quotation conventions, and interest calculation methods are what give rise to seemingly diverse formulas. We have not covered cases where the spot and forward/futures prices do not converge or where the underlying is not storable. These cases are far more complex, and while the cost of carry model might provide a reasonable approximation, it cannot be indiscriminately applied under those conditions.

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<sup>11</sup>If the system worked perfectly, all bonds would be equally preferable for delivery. The conversion factor system, however, is a linear adjustment system, while bond prices are nonlinearly related. Hence, the conversion factor system does not equalize all bonds. One bond in particular is preferable, and it is usually called the *cheapest-to-deliver*. The futures price will behave as though this bond is the one that will be delivered. Hence, this bond is considered to be the underlying bond. Over time, however, another bond could emerge as the cheapest-to-deliver bond, and the futures contract will then behave as though that bond is the underlying bond.