

TEACHING NOTE 97-02:
OPTION PRICING USING FINITE DIFFERENCE METHODS

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Under the appropriate assumptions, the price of an option is given by the solution, $c(S,t)$, to the familiar partial differential equation

$$\frac{\partial c(S_t, t)}{\partial S_t} r S_t + \frac{\partial c(S_t, t)}{\partial t} + \frac{1}{2} \frac{\partial^2 c(S_t, t)}{\partial S_t^2} S_t^2 \sigma^2 = rc(S_t, t).$$

For standard European options and certain other options, a closed form solution is possible. In the case of European options, the boundary condition $c(S_T, T) = \text{Max}(0, S_T - X)$ where X is the exercise price in the well-known Black-Scholes model. In some cases, however, no closed-form solution can be derived. Although the binomial model can normally be used, the finite difference method is known to be slightly more efficient, although the advantages are fewer the faster the computer.

In this note, I illustrate how to derive the simple European option pricing formula using both the explicit and implicit finite difference methods. The approach taken is the log transform method as suggested by Brennan and Schwartz (1978), which is known to provide a stable solution.¹

We begin by defining $y = \ln S$ and $w(y,t) = c(S,t)$ is the price of the call at time t .² This is the price of the call in terms of the transformed asset price and time. In other words, we price the call in

$$\begin{aligned} \frac{\partial c}{\partial S} &= \frac{\partial w}{\partial y} e^{-y} \\ \frac{\partial^2 c}{\partial S^2} &= \left[\frac{\partial^2 w}{\partial y^2} - \frac{\partial w}{\partial y} \right] e^{-2y} \\ \frac{\partial c}{\partial t} &= \frac{\partial w}{\partial t}. \end{aligned}$$

terms of the log of the asset price and time t . Dropping the y and t notations we have
 Substituting these results into the partial differential equation, we obtain

¹An unstable solution can result in negative option prices for certain asset prices and times to expiration.

²The time subscript on S is dropped except where necessary.

$$\frac{1}{2} \sigma^2 \frac{\partial^2 w}{\partial y^2} + \left[r - \frac{1}{2} \sigma^2 \right] \frac{\partial w}{\partial y} + \frac{\partial w}{\partial t} - rw = 0.$$

To solve this equation for w , we express it as a difference equation,

$$\frac{1}{2} \sigma^2 \frac{\Delta^2 w}{\Delta y^2} + \left[r - \frac{1}{2} \sigma^2 \right] \frac{\Delta w}{\Delta y} + \frac{\Delta w}{\Delta t} - rw = 0.$$

We then partition a reasonable range of the log of the asset price into finite intervals. For example, the minimum asset price is zero and the maximum is infinity. Suppose we consider only the following prices as feasible: $0, \Delta y, 2\Delta y, \dots, y - 2\Delta y, y - \Delta y, y, y + \Delta y, y + 2\Delta y, \dots, \infty$. When we implement the technique, we shall need to make Δy as small as possible and we must choose a finite maximum asset price. In addition $\ln 0$ is left undefined so we let the minimum $\ln S$ be very close to but not equal to zero.³ Here we specify $\min \ln S$ as ϵ .

Letting $\tau = T - t$, the time to expiration, we partition the time remaining in the option's life into discrete intervals equaling $\tau, \tau - \Delta t, \tau - 2\Delta t, \dots, 2\Delta t, \Delta t, 0$.

These gradations of time and asset price can be arranged on a grid.

lnS	Time to Expiration						
	τ	$\tau - \Delta t$	$\tau - 2\Delta t$	\bullet	$2\Delta t$	Δt	0
∞	\bullet	\bullet	\bullet	\bullet	\bullet	\bullet	\bullet
\bullet	\bullet	\bullet	\bullet	\bullet	\bullet	\bullet	\bullet
$y + 2\Delta y$	\bullet	\bullet	\bullet	\bullet	\bullet	\bullet	\bullet
$y + \Delta y$	\bullet	\bullet	\bullet	\bullet	\bullet	\bullet	\bullet
y	\bullet	\bullet	\bullet	\bullet	\bullet	\bullet	\bullet
$y - \Delta y$	\bullet	\bullet	\bullet	\bullet	\bullet	\bullet	\bullet
$y - 2\Delta y$	\bullet	\bullet	\bullet	\bullet	\bullet	\bullet	\bullet
\bullet	\bullet	\bullet	\bullet	\bullet	\bullet	\bullet	\bullet
$2\Delta y$	\bullet	\bullet	\bullet	\bullet	\bullet	\bullet	\bullet
Δy	\bullet	\bullet	\bullet	\bullet	\bullet	\bullet	\bullet
ϵ	\bullet	\bullet	\bullet	\bullet	\bullet	\bullet	\bullet

³Technically S should probably be no less than 1. Otherwise $\ln S$ would be negative.

Each dot corresponds to an option price associated with the log of the asset price in the given row and the time to expiration in the given column.

Some of the information in the grid is already known. For example, we know the following boundary conditions.

If the asset price is zero, the call is worthless regardless of the time to expiration:

$$c(0,t) = 0 \text{ for all } t.$$

In terms of $\ln S$, we have $\ln S = \varepsilon$ and with ε very close to zero, we specify this condition as

$$w(\varepsilon,t) = 0 \text{ for all } t.$$

This permits us to fill in zeroes along the bottom row.

When $S \rightarrow \infty$, the first derivative of the call price with respect to the asset price is 1:

$$\lim_{S \rightarrow \infty} \partial c(S,t)/\partial S = 1 \text{ for all } t.$$

Since $\partial c(S,t)/\partial S = (\partial w/\partial y)e^{-y}$, we have

$$\frac{\partial w(y,t)}{\partial y} = e^y = S \text{ for all } t \text{ when } \ln S \rightarrow \text{infinity}.$$

This means that once we know the second highest call value, we can obtain the highest value by adding Δye^y to it.

The intrinsic value at expiration is given as

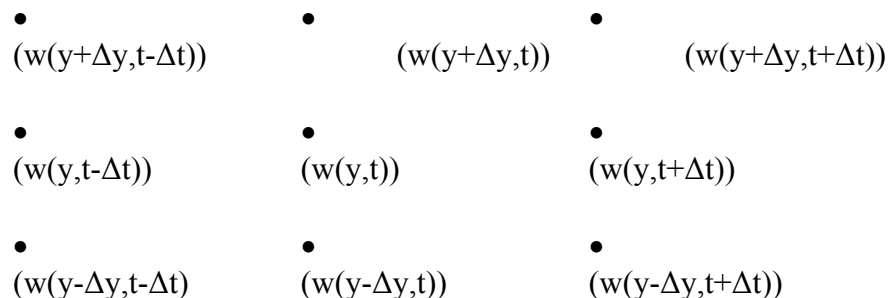
$$c(S,0) = \max(0, S - X) \text{ for all } S.$$

In terms of y , this is simply

$$w(y,0) = \max(0, e^y - X) \text{ for all } y.$$

Thus, we can fill in the entire right column.

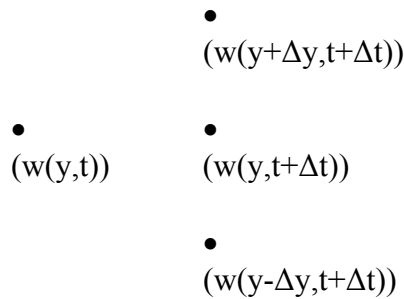
In what follows, it may be useful to visualize the following section from the grid.



Each item indicated by the dot and the expression in parentheses below is the price of a particular option with the log of the asset at the level indicated and the time to expiration as shown.

The Explicit Finite Difference Method

This method solves for $w(y,t)$ in terms of the known values of $w(y+\Delta y,t+\Delta t)$, $w(y,t+\Delta t)$ and $w(y-\Delta y,t+\Delta t)$, which as the diagram above indicates, are the option prices one time step forward and one asset price up, the current asset price and one asset price down. The three prices one time step ahead are assumed to be already known. We show later how their values are indeed already known.



We can obtain finite difference estimates of the partial derivatives as follows. $\Delta w(y,t)/\Delta t$ is approximately,

$$\frac{\Delta w(y,t)}{\Delta t} = \frac{w(y,t+\Delta t) - w(y,t)}{\Delta t}.$$

The differential $\Delta w/\Delta y$ can be estimated as the forward difference,

$$\frac{\Delta w(y,t)}{\Delta y} = \frac{w(y+\Delta y,t+\Delta t) - w(y,t+\Delta t)}{\Delta y},$$

or the backward difference,

$$\frac{\Delta w(y,t)}{\Delta y} = \frac{w(y+\Delta y,t+\Delta t) - w(y-\Delta y,t+\Delta t)}{2\Delta y}.$$

We typically average these two estimates to obtain

$$\frac{\Delta w(y,t)}{\Delta y} = \frac{w(y,t+\Delta t) - w(y-\Delta y,t+\Delta t)}{\Delta y}.$$

The second partial differential can be estimated as

Substituting these estimates of the partial differentials into the difference equation and

$$\begin{aligned}\frac{\Delta^2 w(y, t)}{\Delta y^2} &= \frac{\frac{w(y + \Delta y, t + \Delta t) - w(y, t + \Delta t)}{\Delta y} - \frac{w(y, t + \Delta t) - w(y - \Delta y, t + \Delta t)}{\Delta y}}{\Delta y} \\ &= \frac{w(y + \Delta y, t + \Delta t) - 2w(y, t + \Delta t) + w(y - \Delta y, t + \Delta t)}{\Delta y^2}.\end{aligned}$$

rearranging gives us

$$w(y, t) = \frac{\omega_1 w(y - \Delta y, t + \Delta t) + \omega_2 w(y, t + \Delta t) + \omega_3 w(y + \Delta y, t + \Delta t)}{1 + r\Delta t},$$

where

$$\begin{aligned}\omega_1 &= \left[\frac{1}{2} (\sigma/\Delta y)^2 - \frac{1}{2} (r - \sigma^2/2)/\Delta y \right] \Delta t \\ \omega_2 &= 1 - (\sigma/\Delta y)^2 \Delta t \\ \omega_3 &= \left[\frac{1}{2} (\sigma/\Delta y)^2 + \frac{1}{2} (r - \sigma^2/2)/\Delta y \right] \Delta t.\end{aligned}$$

In other words, the option price at t is a discounted weighted average of the next three possible option prices at $t + \Delta t$, where the weights are given as above. These weights sum to unity and are the risk neutral/equivalent martingale probabilities of the three log asset prices $y + \Delta y$, y , and $y - \Delta y$ at $t + \Delta t$. This technique is, therefore, a risk neutralized trinomial method.

Filling in the full grid with the option prices is simple. As noted previously, the right-most column is the set of option prices at expiration and is already known. Using the right-most column to provide the three known option prices, we can then solve for the second-to-right-most column of option prices. Other missing prices in the top-most row can be filled in by knowing the second-to-top-most row, as noted earlier. The prices in the bottom-most row are, as noted earlier, zero. The process then continues leftward until the left-most column is filled in.

When the entire grid is filled in, the current option price can be read from the row corresponding to the log of the current asset price in the left-most column.⁴

⁴The option's delta, gamma and theta can be estimated by using the formulas for $\partial w/\partial y$, $\partial^2 w/\partial y^2$ and $\partial w/\partial t$.

$$\begin{aligned}\frac{\Delta^2 w(y,t)}{\Delta y^2} &= \frac{\frac{w(y+\Delta y,t) - w(y,t)}{\Delta y} - \frac{w(y,t) - w(y-\Delta y,t)}{\Delta y}}{\Delta y} \\ &= \frac{w(y+\Delta y,t) - 2w(y,t) + w(y-\Delta y,t)}{\Delta y^2}.\end{aligned}$$

The time differential is estimated the same way as in the explicit finite difference method.

Now we substitute these values into the difference equation to obtain

$$\begin{aligned}\frac{1}{2}\sigma\left(\frac{w(y+\Delta y,t) - 2w(y,t) + w(y-\Delta y,t)}{\Delta y^2}\right) \\ + (r - \sigma^2/2)\left(\frac{w(y+\Delta y,t) - w(y-\Delta y,t)}{2\Delta y}\right) \\ + \frac{w(y,t+\Delta t) - w(y,t)}{\Delta t} - rw(y,t) = 0.\end{aligned}$$

Solving for $w(y,t+\Delta t)$ gives

$$w(y,t+\Delta t) = \omega_1 w(y-\Delta y,t) + \omega_2 w(y,t) + \omega_3 w(y+\Delta y,t),$$

where

$$\begin{aligned}\omega_1 &= \frac{1}{2} \frac{(r - \sigma^2/2)\Delta t}{\Delta y} - \frac{1}{2} \frac{\sigma^2 \Delta t}{\Delta y^2} \\ \omega_2 &= 1 + \frac{\sigma^2 \Delta t}{\Delta y^2} + r\Delta t \\ \omega_3 &= -\frac{1}{2} \frac{(r - \sigma^2/2)\Delta t}{\Delta y} - \frac{1}{2} \frac{\sigma^2 \Delta t}{\Delta y^2}.\end{aligned}$$

Note how the equation above solves for the prices $w(y+\Delta y,t)$, $w(y,t)$ and $w(y-\Delta y,t)$ in terms of the known value $w(y,t+\Delta t)$. With three unknowns there must be three equations. Thus, the solution is obtained by solving simultaneously an entire column of option prices. This starts at the next-to-right-most column and works leftward.

The implicit finite difference method is generally considered the better approach but it is slightly more difficult to implement, requiring as it does, solving simultaneous equations. The

Crank-Nicholson method is also widely used. It essentially averages the explicit and implicit methods.

Finite Difference Put Option Pricing

If the options are puts, then the procedure is exactly the same but the following three conditions replace the corresponding conditions for calls.

$$\begin{aligned}w(\varepsilon, t) &= Xe^{-rt} \text{ when } \varepsilon \rightarrow 0 \\w(y, t) &= 0 \text{ for all } t \text{ when } y \rightarrow \text{infinity} \\w(y, 0) &= \max(0, X - e^y) \text{ for all } y.\end{aligned}$$

Dividends and Early Exercise

If the asset is a stock and there are dividends, the procedure is slightly more complex. If there is a continuous, constant yield of δ , the first term in the partial differential equation is changed slightly with $r - \delta$ replacing r . Then S is redefined as $Se^{-\delta t}$. If there are discrete dividends, the partial differential equation is the same as the no-dividend case but the asset price is redefined as the asset price less the present value of the dividends.

If the options are American calls, then the possibility of early exercise must be considered. At each node you must determine if the asset is worth more exercised early. You simply calculate the value of the option the ordinary way and then calculate the intrinsic value, $\max(0, S + PV(D) - X)$ where $PV(D)$ is the present value of all remaining dividends during the life of the option. If the call is worth more exercised, then replace the calculated value of the option with the intrinsic value. This will occur with high dividends, low time value, low volatility, and/or a low interest rate.

If the options are American puts, dividends are not required to justify early exercise. The same procedure as described above for calls is followed, with of course appropriate substitution of the put boundary conditions in the top-most and bottom-most rows and right-most columns.

References

The first paper to recognize that option prices could be obtained with a finite difference solution to the partial differential equation was

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An improvement to the technique is presented in

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A comparison of alternative finite difference and binomial methods is in

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Other useful readings are

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