

**TEACHING NOTE 97-03:
THE VASICEK TERM STRUCTURE MODEL**

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One of the earliest stochastic models of the term structure was developed by Vasicek (1977). His model is based on the evolution of an unspecified short-term interest rate. Although the model is developed in a fairly general framework, it is most frequently applied in a more specific form that is presented here. The Vasicek model has many advantages, though it also has some shortcomings. The model assumes that the short rate follows the stochastic process

$$dr = a(b - r)dt + \sigma dz.$$

Here dz is a standard Wiener process and r is the current level of the interest rate. The parameter b is the long run normal interest rate. The model exhibits *mean reversion*, which means that if the interest rate is above the long run mean ($r > b$), then the coefficient a (>0) makes the drift become negative so that the rate will be pulled down in the direction of r . Likewise, if the rate is less than the long run mean, $r < b$, then the coefficient a makes the drift become positive so that the rate will be pulled up in the direction of r . The coefficient a is, thus, the speed of adjustment of the interest rate towards its long run normal level. This feature is particularly attractive because without it, interest rates could drift permanently upward the way stock prices do and this is simply not observed in practice. This particular type of stochastic process is referred to as an *Ornstein-Uhlenbeck process*.

In the Vasicek model the risk is captured by assuming that the market price of interest rate risk, $(\mu - r)/\sigma = \lambda$, is constant across the term structure. This assumption is essentially the same as the no-arbitrage/equivalent martingale assumption and permits pricing in a risk-neutral framework. This results in the following formula for the price of a \$1 face value zero coupon bond,

$$P(t, T) = A(t, T)e^{-B(t, T)r}$$

$$\text{Where } B(t, T) = \frac{1 - e^{-a(T-t)}}{a}$$

$$\text{and } A(t, T) = \exp\left[\frac{B(t, T) - (T - t)(a^2b - \sigma^2 / 2)}{a^2} - \frac{\sigma^2 B(t, T)}{4a}\right].$$

In the Vasicek model, the interest rate is normally distributed and the expectation and variance of the future interest rate at time T are given as

$$E[R(T)|r(0)] = r(0)e^{-a(T-t)} + b(1 - e^{-a(T-t)})$$

$$\text{Var}[R(T)|r(0)] = \frac{\sigma^2(1 - e^{-2a(T-t)})}{2a}.$$

One unfortunate consequence of a normally distributed interest rate is that it is possible for the interest rate to become negative. Taking the limit of the expected rate and variance when $T \rightarrow \infty$ shows, however, that as long as $a > 0$, the expectation will converge to b and the variance will converge to $\sigma^2/2a$.

A special case of the Vasicek model is when $a = 0$. This model was first developed by Dothan (1977). In that case $B(t,T) = T - t$, $A(t,T) = \exp[\sigma(T - t)^3/6]$.

Jamshidian (1989) has shown that the price of a European call option on a pure discount bond in the Vasicek model can be easily derived. Let the bond mature at time s and the option mature at time T , where $T < s$. With X as the strike price of the option, the call price will be given as

$$c(P(t,s),T) = P(t,s)N(h) - XP(t,T)N(h - \sigma_p)$$

$$\text{with } h = \frac{1}{\sigma_p} \ln \left[\frac{P(t,s)}{P(t,t)X} \right] + \frac{\sigma_p}{2}$$

$$\sigma_p = v(t,T)B(T,s)$$

$$v(t,T)^2 = \frac{\sigma^2(1 - e^{-2a(T-t)})}{2a}.$$

A put is worth $XP(t,t)N(-h + \sigma_p) - P(t,s)N(-h)$. For the Dothan model where $a = 0$, we have $v(t,T) = \sigma\sqrt{T-t}$, $\sigma_p = \sigma(s-t)\sqrt{T-t}$.

Jamshidian has also shown that an option on a coupon bond can also be easily valued. On the option's expiration date, the bond will have n cash flows yet to come, with each cash flow expressed as $c_i, i=1, n$. Let r^* be the value of the short term rate that makes the coupon bond be at-the-money at expiration. Let X_i be the value at the option's expiration of a zero coupon bond at time s_i when $r = r^*$. The value r^* can be found by trial and error. The payoff of an option on the coupon bond can be shown to be

$$\sum_{i=1}^n c_i \max[0, P(r, T, s_i) - X_i],$$

or in other words, the sum of the payoffs of the options on the individual discount bonds that make up the coupon bond. Consequently the price of an option on a coupon bond is the sum of the prices of options on each of the cash flows on the bond.

The prices of American options in the Vasicek model must be estimated by a numerical procedure. Hull and White (1990b) employ a trinomial method that allows for different branching structures. The method permits the user to solve for different probabilities at each node, which uphold the constraint that the probabilities must sum to one and that they must guarantee that the interest rate will be normally distributed with mean and variance correctly defined. This procedure provides the prices of the unit discount bonds and the option prices by backwards iteration.

There are several weaknesses of the Vasicek model. While the model is arbitrage-free in the sense that no bond or option prices produced by the model will permit arbitrage, it is, nonetheless not arbitrage-free in the context of actual market prices. This is because the model produces a term structure as an output but does not accept the current term structure as an input. A dealer offering bonds and options for sale would have to offer prices consistent with the current term structure. Otherwise no one would trade with it because its prices were not competitive or everyone would trade with it because its prices permit arbitrage. A dealer using the Vasicek model would generate prices that are inconsistent with the current term structure and this could permit other dealers and counterparties to arbitrage against the dealer. In essence what it amounts to is that the dealer using the Vasicek model is claiming to be offering the correct prices but everyone else claims another set of prices as correct.

Another limitation of the Vasicek model is that it is a one factor model and cannot capture the more complex term structure shifts that occur. Another questionable assumption is that all rates have the same volatility.

Suppose we wish to fit the Vasicek model to a binomial tree. Consider the following tree of spot rates.

		r^{++}
	r^+	
r		r^{+-} r^{-+}
	r^-	
		r^{--}

We would like to know the values of r^+ , r^{++} , r^{+-} , r^- , and r^{--} . In addition, we would like the tree to recombine so that $r^{+-} = r^{-+}$. From TN00-05: Brownian Motion: From Discrete to Continuous time, we know that we create a Wiener process by defining a time step as of unit length and adding or subtraction the volatility. Hence,

$$\begin{aligned} r^{++} &= r^+ + a(b - r^+) - \sigma \\ r^{-+} &= r^- + a(b - r^-) + \sigma. \end{aligned}$$

To get the tree to recombine, we can set these equal to obtain

$$2\sigma = (r^+ - r^-)(1 - a).$$

But by definition, the volatility is

$$\sigma = \frac{r^+ - r^-}{2}.$$

These two statements imply that $a=0$, which takes away the drift and mean reversion, a major feature of Vasicek model. It would be possible to fit a variation of the model, one with non-constant volatility, but that would not technically be the Vasicek model.¹

As noted above, the Vasicek model does not incorporate the current term structure. Hull and White (1990a) have proposed a modification to the model to incorporate the current term structure. They introduce a time varying volatility, drift, and mean reversion parameter:

$$dr = [\theta(t) + a(t)(b - r)]dt + \sigma(t)dz.$$

¹In Chapter 16 of Jarrow-Turnbull (2000), the authors fit a normally distributed model to the current term structure. They do not give the model a name, but it can be viewed as a variation of the Vasicek model, one in which the volatility of the spot interest rate is different at different points in time and one that fits the current term structure. It can also be viewed as a variation of the Ho-Lee model with non-constant volatility but also with the feature of mean reversion.

The drift term, $\phi(t)$, is $a(t)b + \theta(t)$ minus a risk premium, if applicable, and $\theta(t)$ is given by a more complex formula in their paper. The solutions for the bond price and $B(t,T)$ are the same as before but $A(t,T)$ is now

$$A(t,T) = \exp \left[\frac{B(t,T) - (T-t)(a\phi - \sigma^2/2)}{a^2} - \frac{\sigma^2 B(t,T)^2}{4a} \right]$$

for the case where a , ϕ , and σ are constant across time. For the cases when these values are not independent of time, they present equations on pp. 578-580 that show how information from the term structure can be used to produce the input values and, thus, make the model consistent with the current term structure. They also give formulas for options on discount and coupon bonds, which are similar to Jamshidian's formulas.

Hull and White (1993) also propose another variation of the model of the form

$$dr = [\theta(t) - ar]dt + \sigma dz.$$

In this case, a is constant. They demonstrate how a trinomial tree will provide the prices for European and American options. Hull and White's trinomial models fit the current term structure to the model and update the parameters as they step through time. This is viewed as an advantage by them but others believe it is simply a procedure that recalibrates with no real time-dependent structure.

References

The original article is

Vasicek, Oldrich. "An Equilibrium Characterization of the Term Structure." *Journal of Financial Economics* 5 (1977), 177-188.

The following paper provides the option pricing formulas for the Vasicek model and many other similar models.

Jamshidian, Farshid. "An Exact Bond Option Pricing Formula." *The Journal of Finance* 44 (1989), 205-209.

Other related papers are

Dothan, L. Uri. "On the Term Structure of Interest Rates." *Journal of Financial Economics* 6 (1978), 59-69.

Hull, John and Alan White. "One-Factor Interest Rate Models and the Valuation of Interest Rate Derivative Securities." *Journal of Financial and Quantitative Analysis* 28 (1993), 235-254.

Hull, John and Alan White. "Pricing Interest Rate Derivative Securities." *Review of Financial Studies* 3 (1990a), 573-592.

Hull, John and Alan White. "Valuing Derivative Securities Using the Explicit Finite Difference Method." *Journal of Financial and Quantitative Analysis* 25 (1990b), 87-100.