

TEACHING NOTE 97-04:
THE COX-INGERSOLL-ROSS TERM STRUCTURE MODEL

Version date: August 8, 2008

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Cox-Ingersoll-Ross (1985) developed one of the first general equilibrium theories of the term structure of interest rates. Out of that theory came a model for the pricing of zero coupon bonds and derivatives. The CIR model is based on the following stochastic process for the short rate:

$$dr = a(\mu - r)dt + \sigma\sqrt{r}dW(t)$$

The model exhibits mean reversion of the interest rate, causing the rate to be pulled downward when it is above the long run average rate and be pulled upward when it is below the long run average rate. The coefficient a is the speed of this mean reversion, μ is the long run average rate, r is the current rate, and σ is the volatility. The model does provide for pricing in a general equilibrium framework but would require the market price of risk, an unobserved value, defined as $\lambda(t) = \lambda\sqrt{r}/\sigma$. Most applications of the CIR model assume the local expectations hypothesis, which is equivalent to the equivalent martingale/risk neutral assumption. In that case, the value of a \$1 face value pure discount bond is

$$P(t,T) = A(T-t)e^{-B(T-t)r}$$

where $A(T-t) = \left[\frac{2\gamma e^{(a+\gamma)(T-t)/2}}{(\gamma+a)(e^{\gamma(T-t)}-1) + 2\gamma} \right]^{2a\mu/\sigma^2}$

and $B(T-t) = \frac{2(e^{\gamma(T-t)}-1)}{(\gamma+a)(e^{\gamma(T-t)}-1) + 2\gamma}$

$$\gamma = \sqrt{a^2 + 2\sigma^2}.$$

In this model, the interest rate is distributed as a chi-square. This model, like that of Vasicek, does not fit the current term structure but rather provides the current term structure. As a result, its prices, though internally free from arbitrage, would not be consistent with arbitrage-free prices in the market. Consequently, users of this model could suffer arbitrage losses.

CIR give the price of a futures contract expiring at T where the underlying pure discount bond expires at s with $s > T$ as

$$H(t, T, s) = A(s - T) \left[\frac{\rho}{B(s - T) + \rho} \right] \exp \left[\frac{-r \rho B(s - T) e^{-a(T-t)}}{B(s - T) + \rho} \right]$$

$$\text{where } \rho = \frac{2a}{\sigma^2 (1 - e^{-a(T-t)})}$$

and the values $A(s - T)$ and $B(s - T)$ are forward based values, computed as in the formulas above for $A(T - t)$ and $B(T - t)$ by simply substituting the maturity of $s - T$.

A call option with strike X expiring at time s where $s > T$ is given by the following formula,

$$c(t, T, s, X) = P(t, s) \chi_n^2(x_1; v_1, v_2) - XP(t, T) \chi_n^2(x_2; v_1, v_3)$$

$$x_1 = 2r^* [\phi + \theta + B(s - T)], \quad x_2 = x_1 - B(s, T) 2r^*$$

$$v_1 = 4a\mu / \sigma^2, \quad v_2 = \frac{2\phi^2 r e^{\gamma(T-t)}}{\phi + \theta + B(s - T)}, \quad v_3 = \frac{2\phi^2 r e^{\gamma(T-t)}}{\phi + \theta}$$

$$\phi = \frac{2\gamma}{\sigma^2 (e^{\gamma(T-t)} - 1)}, \quad \theta = (a + \gamma) / \sigma^2$$

$$r^* = \log \left[\frac{A(s - T)}{X} \right] / B(s - T)$$

The put price for this (and most of the other models) can be obtained from put-call parity using the value $P(t, s)$ as the value of the underlying and $P(t, T)$ as the discount function on the exercise price.

American option prices must be calculated using a tree. One suggested tree that simplifies this model is based on an expansion of the variable $x = 2\sqrt{r}$ while another suggested approach divides this by the standard deviation (Tian (1993)). We illustrate that approach here, following the material in Ritchken (1996, Ch. 23). This approach leads to a stable tree, i.e., probabilities in the interval $[0, 1]$.

Consider the following model. We let the time step increment be .2 years, the current rate be 10 %, $\sigma = 10$ %, $\alpha = .6$, and $\mu = 10$ %. In other words, we start off at a rate of 10 % with a volatility of 10 %. The long run mean rate is also 10 % and the rate pulls back toward the mean at a rate of 60

% . We then derive the tree for the transformed variable $x = 2\sqrt{r}$. With a starting value of $r = 0.1$, we have $x = 6.3245$. The variable x will follow the stochastic process, $dx = \mu(x)dt + dz$. The binomial tree will expand according to the following rule: $x^+ = x + (\Delta t)^2$ and $x^- = x - (\Delta t)^2$. Following this rule gives us a tree for x that expands as follows:

Time 0	Time 1	Time 2	Time 3	Time 4	Time 5
					8.5606
				8.1134	
			7.6620		7.6662
		7.2190		7.2189	
	6.7718		6.7718		6.7718
6.3246		6.3246		6.3246	
	5.8773		5.8773		5.8773
		5.4301		5.4301	
			4.9829		4.9829
				4.5357	
					4.0885

Transforming this number gives us the continuously compounded rate on one-period default-free bonds. That rate will be $r = \sigma^2 x^2 / 4$. Using the values in the tree above gives us the tree of one-period interest rates as follows:

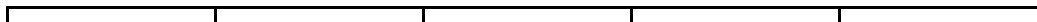
Time 0	Time 1	Time 2	Time 3	Time 4	Time 5

					18.3 %
				16.5 %	
			14.7 %		14.7 %
		13.0 %		13.0 %	
	11.5 %		11.5 %		11.5 %
10 %		10 %		10 %	
	8.6 %		8.6 %		8.6 %
		7.4 %		7.4 %	
			6.2 %		6.2 %
				5.1 %	
					4.2 %

The probability of an up-move is found as

$$p(r) = \frac{a(\mu - r)\Delta t + (r - r^-)}{(r^+ - r^-)}$$

indicating that this probability changes with the level of r and is determined, among other things, by the next two possible rates, r^+ and r^- . The probability of a down move is simply $1 - p(r)$. With these probabilities changing in the tree, we should construct a tree of up probabilities. For the first one, note that $a = .6$, $\mu = 0.1$, $r = 0.1$, $r^+ = 0.115$ and $r^- = 0.086$. Then the initial up probability is $[0.6(0.1 - 0.1)(0.2) + (0.1 - 0.086)] / (0.115 - 0.086) = 0.482$. The remaining probabilities are similarly found and the tree of probabilities is presented below (note that we automatically lose the final period):



Time 0	Time 1	Time 2	Time 3	Time 4
				0.273
			0.321	
		0.372		0.372
	0.426		0.426	
0.482		0.482		0.482
	0.543		0.543	
		0.609		0.609
			0.682	
				0.763

We can now find the one-period default-free discount bond prices by using the above rates multiplied by the time step, 0.2. In other words the current price is $e^{-0.1(0.2)} = 0.980$. As another example, look ahead to the price after the rate has gone up two periods and down one. The rate will then be 11.5 %. The bond price is, therefore, $e^{-0.115(0.2)} = 0.977$.

We can also price bonds of greater than one-period maturity. In that case the forward rate for one-period bonds, one period hence will equal the expected spot rate. For example, suppose we are in time period 3 and the spot rate has gone up two times and down once. We are at the node where the one period bond price is 0.977, the one period rate is 11.5 % and the probability of an up move is 0.426. If the rate moves up, the next bond price will be 0.974 and if the rate moves down the next bond price will be 0.980. The price of a two-period bond will equal the one-period bond price times the forward price. The forward price can be found as the expected future spot price, which is given as $0.974(0.426) + 0.980(1 - 0.426) = 0.977$. Thus, the two-period bond price can be found as $0.977(0.977) = 0.955$. In a similar manner we can find the prices of bonds of greater maturities. For example, a three period bond price will equal the price of a two-period bond times the one-period

ahead forward rate, which can be derived as the expected future spot rate. Keep in mind, however, that the forward price equals the expected future spot price only under the local expectations hypothesis, which is the same as the equivalent martingale hypothesis/no-arbitrage hypothesis. It means that the expected return over the shortest holding period, here one time step, is the same for all bonds.

We can derive the price of a bond of any maturity an alternative way. We start at time 5 and price the one-period bonds as the probability-weighted discounted (at the appropriate one-period rate) average of the next two possible prices. After filling in all time 5 prices of one-period bonds, we step back to time 4 and recognize that 2-period bonds at time 4 will evolve into 1-period bonds at time 5. We then simply take the probability-weighted discounted average of the next two 1-period bond prices at time 5. For example, at the top-most node at time 5 we have a one-period rate of 18.3 %, meaning that the one-period bond price is $e^{-0.183(0.2)} = 0.964$. At the node just below, the one-period bond price is $e^{-0.147(0.2)} = 0.971$. Stepping back to the top-most node at time 4, let us price a two-period bond at that time. We see that this two-period bond price will evolve into a one-period bond price of either 0.964, with (equivalent martingale) probability 0.273, or 0.971, with (equivalent martingale) probability $1 - 0.273 = 0.727$. The expected price one period later is, thus, $0.964(0.273) + 0.971(0.727) = 0.969$. We then discount this back from period 5 to period 4 at the one-period rate in the period four cell, which is 16.5 %. So we have $0.969e^{-0.165(0.2)} = 0.9375$. This procedure is then repeated throughout the tree.

Hull and White (1990) have proposed a modification of the CIR model that permits input of the current term structure. They introduce a time varying volatility, drift and mean reversion parameter:

$$dr = [\theta(t) + a(t)(b - r)]dt + \sigma(t)\sqrt{r}dz.$$

The drift term $\varphi(t)$ is given as $a(t)b + \theta(t)$ minus a risk premium, if applicable, and $\theta(t)$ is given by a more complex formula in the paper. The solution for the bond price is the same but the following are slightly different:

$$\text{where } A(T-t) = \left[\frac{2(e^{(\gamma+\psi)(T-t)/2} - 1)}{(\gamma + \psi)(e^{\gamma(T-t)} - 1) + 2\gamma} \right]^{2\phi/\sigma^2}$$

$$\text{and } B(T-t) = \frac{2(e^{(\gamma+\psi)(T-t)/2} - 1)}{(\gamma + \psi)(e^{\gamma(T-t)} - 1) + 2\gamma}$$

$$\gamma = \sqrt{\psi^2 + 2\sigma^2},$$

where many of the parameters are time independent. For time dependent parameters and the case of American options, they recommend a trinomial tree.

Hull and White (1993) also propose another variation of the form

$$dr = [\theta(t) - ar]dt + \sigma\sqrt{r}dz$$

where a is constant. Again, they use a trinomial tree to calculate these values.

References

The original article is

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