TEACHING NOTE 97-05:
THE BIVARIATE NORMAL PROBABILITY DISTRIBUTION

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In Teaching Note 97-01, we learned about the normal probability distribution. Suppose now that we have two normally distributed random variables, X and Y. The expected values are μ_X and μ_Y and the standard deviations are σ_X and σ_Y. We let the conditional expected value of Y and X be linearly related, as evidenced by the fact that

$$E[Y|x] = \mu_Y + \rho(\sigma_Y/\sigma_X)(x - \mu_X),$$

where ρ is the correlation between Y and X. This statement simply says that if the value of X is known, the expected value of Y is given by the right-hand side expression. This expectation of Y is called the conditional expected value of Y, given x. The terms μ_Y and μ_X are the unconditional expected values. They are our best estimates of the expected values of Y or X, given that we know nothing about the value of the other. If X and Y are linearly related, then the correlation between X and Y can be used to make a better prediction of Y, given that we know the current value of X, which is x. If Y and X are related in this manner, then the joint distribution of Y and X is bivariate normal. The conditional variance of Y is related to its unconditional variance by the formula[1]

$$\sigma^2_{Y|X} = \sigma^2_Y\left(1-\rho^2\right).$$

The probability density function for the bivariate normal is

$$f(x,y,\rho) = \frac{1}{2\pi\sigma_x\sigma_y\sqrt{1-\rho^2}}$$

$$\exp\left[-\frac{1}{2}\left(\frac{((x-\mu_x)/\sigma_x)^2 - 2\rho((x-\mu_x)/\sigma_x)((y-\mu_y)/\sigma_y) + ((y-\mu_y)/\sigma_y)^2}{1-\rho^2}\right)\right].$$

The distribution function or cumulative bivariate normal probability is

$$\Pr ob\left(X \le x,\ Y \le y|\rho\right) = \frac{1}{\sigma_X\sigma_Y}\int_{-\infty}^{\frac{x-\mu_X}{\sigma_X}}\int_{-\infty}^{\frac{y-\mu_Y}{\sigma_Y}} \frac{1}{2\pi\sqrt{1-\rho^2}}\exp\left[-\frac{1}{2}\left(\frac{s^2-2\rho st+t^2}{1-\rho^2}\right)\right]dsdt\ .$$

[1]For a brief review of the relationships between the joint, conditional and marginal probabilities, see the Appendix to this Teaching Note.

Since each variable X and Y is individually normally distributed, each can be transformed or normalized into a standard normal random variable, which we shall call z_1 and z_2, by the relationships,

$$z_1 = \frac{x - \mu_X}{\sigma_X}, \quad z_2 = \frac{y - \mu_Y}{\sigma_Y}.$$

The standard normal bivariate density is then

$$f(z_1, z_2, \rho) = \frac{1}{2\pi\sqrt{1-\rho^2}} \exp\left[-\frac{1}{2}\left(\frac{z_1^2 - 2\rho z_1 z_2 + z_2^2}{1-\rho^2}\right)\right].$$

The figure on page 4 illustrates the standard normal bivariate density in three-dimensional space.

The following relationships are useful when dealing with the bivariate normal probability distribution. Let N(x) be the (univariate) normal probability for a variable X and $N_2(x,y;\rho)$ be the bivariate normal probability for the variables X and Y, which can be normalized or not. Then

$$N_2(x,y;\rho) = N_2(y,x;\rho)$$

$$N_2(-x,y;\rho) - N_2(x,y;-\rho) = N(y)$$

$$N_2(x,y;\rho) - N_2(-x,-y;\rho) = N(x) + N(y) - 1.0.$$

Computation of the bivariate normal probability is quite difficult, but an analytic approximation developed by Drezner (1978) is often used and gives a high degree of accuracy. You can obtain an Excel spreadsheet, probcalc.xls, that calculates univariate and bivariate normal probabilities, along with the t-, F-, binomial and χ^2 (chi-square) probabilities, from the course home page.

Using that spreadsheet let us work a problem involving bivariate normal random variables. Let x = 0.74, y = -1.13 and Δ = .32. We wish to know Prob($X \leq 0.74$, $Y \leq -1.13|.32$). The univariate probabilities, which are easily obtained from Excel's =normsdist() function, are N(0.74) = 0.7704 and N(-1.13) = 0.1292. The bivariate normal probability is $N_2(0.74,-1.13;.32)$ = 0.1171. Let us check out the above relationships:

$$N_2(0.74,-1.13;.32) = 0.1171 = N(-1.13,0.74;0.32) = 0.1171$$

$$N_2(-0.74,-1.13;0.32) = .0529; N_2(0.74,-1.13;-0.32) = 0.0763; 0.0529 + 0.0763 = 0.1292$$

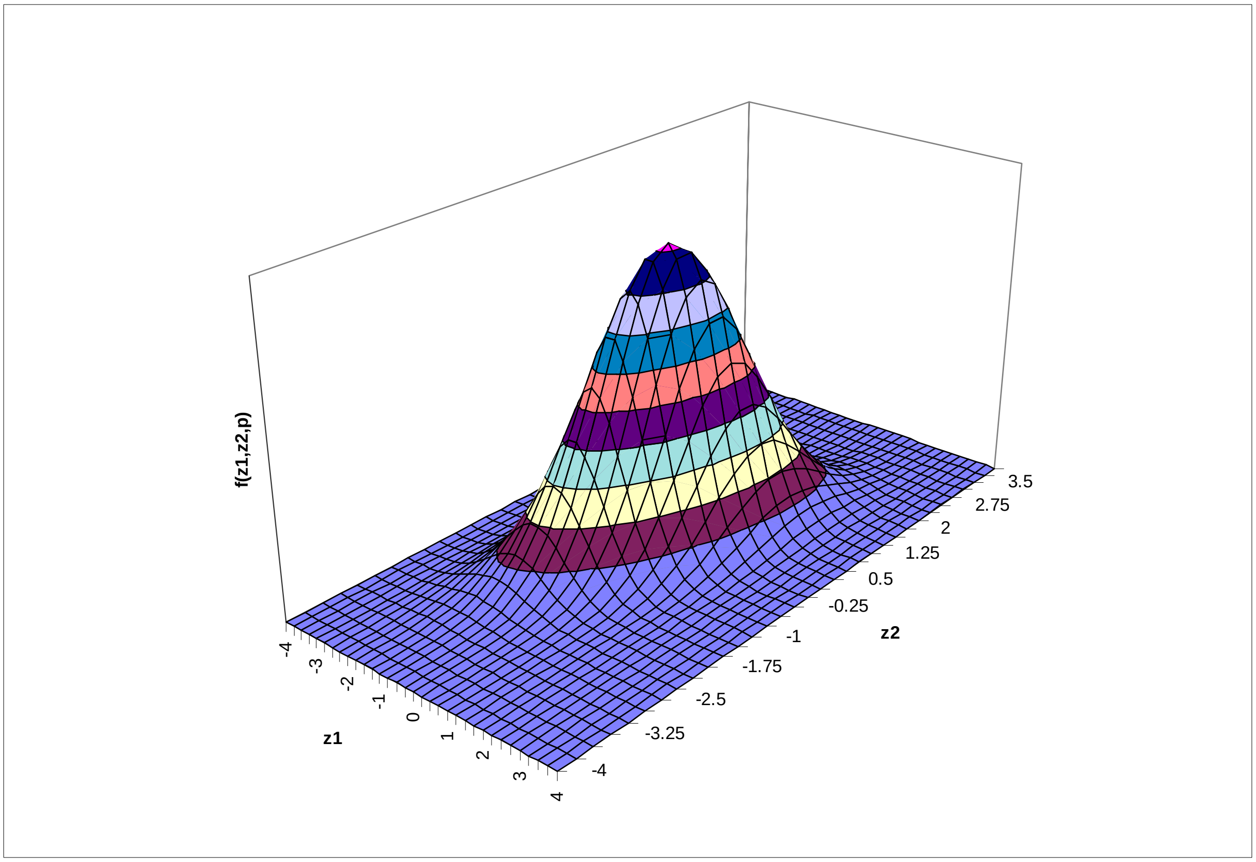
$$= N(-1.13)$$

$$N_2(0.74,-1.13,0.32) = 0.1171;\ N(-0.74,1.13;0.32) = 0.2175;\ 0.1171 - 0.2175 = -0.1004$$

$$N(.74) = 0.7704;\ N(-1.13) = 0.1292;\ 0.7704 + 0.1292 - 1.0 = -0.1004.$$

Some special cases are worth nothing. If either of the values x or y is infinite, then the bivariate probability reverts to the univariate probability. For example, $\text{Prob}(X \leq x, Y \leq \infty|\rho) = \text{Pr}(X \leq x)$. This is because the condition that $Y \leq \infty$ has no effect because its probability is 1.0. This, of course, also holds if the variables are reversed. If $\rho = 0$, then the bivariate probability is the product of the two univariate probabilities of X and Y, i.e., $\text{Prob}(X \leq x, Y \leq y|\rho=0) = N(x)N(y)$. A few other special relationships hold when $\rho = 1$, but this is so rarely observed that I simply refer you to Abramowitz and Stegun (1972).

The bivariate normal probability generalizes into a multivariate normal probability. In finance one occasionally sees the trivariate normal probability and there are techniques for estimating it, when involve simplification of the relationships between univariate, bivariate and trivariate densities. For the most part, however, computation of these high order integrals is extremely time consuming.



References

Most advanced texts in probability theory cover the bivariate normal probability reasonably well. In addition, see

Abramowitz, M. and I. Stegun. *Handbook of Mathematical Functions* (New York: Cover Publications, 1972), Ch. 26.

Divgi, D. R. "Calculation of Univariate and Bivariate Normal Probability Functions." *Annals of Statistics* 7 (1979), 903-919.

Drezner, Z. "Computation of the Bivariate Normal Integral." *Mathematics of Computation* 32 (January, 1978), 277-279.

Drezner, Z. and G. O. Wesolowsky. "The Computation of the Bivariate Normal Integral." *Journal of Statistical Computation and Simulation* 35 (1990), 101-107.

Vasicek, O. F. "A Series Expansion for the Bivariate Normal Integral." *The Journal of Computational Finance* 1 (1998), 5-10.

Papers useful for evaluating higher order integrals are

Curnow, R. N. and C. W. Dunnett. "The Numerical Evaluation of Certain Multivariate Normal Integrals." *Annals of Mathematical Statistics* 33 (June, 1962), 571-579.

Dunnett, C. W. and R. A. Lamm. "Some Tables of the Multivariate Normal Probability Integral with Correlation Coefficients 1/3: Abstract." *Mathematics of Computation* 14 (1960), 290.

Plackett, R. L. "A Reduction Formula for Normal Multivariate Integrals." *Biometrika* 41 (1954), 351-360.

Steck, G. P. "A Table for Computing Trivariate Normal Probabilities." *Journal of the Royal Statistics Society. Series B* 20 (1958), 373-378.

For a treatment of the bivariate normal distribution in option pricing, see my own paper:

Chance, D. M. and S. Agca. "Speed and Accuracy Comparisons of Bivariate Normal Probability Distribution Approximations for Option Pricing." *The Journal of Computational Finance* 6 (2003), 61-96.

APPENDIX: REVIEW OF JOINT, MARGINAL AND CONDITIONAL PROBABILITY

Consider the following information. A sample of 100 people, 55 female and 45 male, is collected and examined for the frequency of brown hair. Of the 55 females, 31 have brown hair and 24 have some other color. Of the 45 males 28 have brown hair and 17 have some other color. If this is a reliable sample, what is the probability that a person selected at random will have brown hair? This is the unconditional probability, sometimes called the marginal probability, and it is 0.59, given that 59 out of 100 have brown hair.

Now suppose I told you that I had selected a male. Then what would be the probability that the subject has brown hair? This is the conditional probability, specifically the probability that the subject would have brown hair, given that it is a male. Then the answer would be 28/45 = 0.62 because 62 % of the males have brown hair.

Now suppose I selected someone at random and asked you the probability that it would be a female with a hair color other than brown. Out of 100 subjects, 24 are females with a hair color other than brown. So the joint probability would be 0.24.

The joint probability is equal to the conditional probability times the marginal probability of the condition. In our example this means that we want the probability that the subject will be a female with a hair color other than brown and this will be the probability that the subject will be female, given that you know the subject does not have brown hair, times the probability that the subject does not have brown hair. The probability that the subject will be female, given that we know that the subject does not have brown hair is 24/41, since there are 41 people with a hair color other than brown and 24 are female. This is the conditional probability. The probability that the subject does not have brown hair is 41/100 since you know that out of 100 people, 41 do not have brown hair. This is the marginal probability. Multiplying the conditional probability times the marginal probability gives us (24/41)(41/100) = 24/100.

Alternatively, we could have found the conditional probability that the subject will not have brown hair given that we know it is a female, which will be 24/55, times the probability that the subject is female, 55/100, giving us the correct answer of 24/100.