

**TEACHING NOTE 97-15:
PRICING AND VALUATION OF EQUITY SWAPS**

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An equity swap is a transaction in which one party agrees to make to the other a series of payments that are determined by the return on a stock or stock index. The other party, in turn, makes to the first party a series of payments that can be at a fixed rate, a floating rate or the return on another stock or index. An equity swap is designed to replicate the returns that would be earned from buying the stock, which could have been purchased by either selling other stock, borrowing at a fixed rate or borrowing at a floating rate. Alternatively an equity swap can be used to remove exposure to a stock or index from a portfolio, replacing it with some other type of exposure. In recent years equity swaps have also been used to reduce the risk of an executive's large investment in the firm in the form of stock and/or options.

Equity swaps differ in several ways from interest rate swaps. Since a stock return can be negative, the party receiving the return on the stock will have to pay the return on the stock when the stock goes down. Thus, in contrast to an interest rate swap where party A makes an interest payment to party B and party B makes an interest payment to party A, an equity swap can have one party making and one party receiving both payments. For both equity and interest rate swaps, however, the exchange of cash is usually done by netting the amounts owed, so there is only one cash amount transferred from one party to the other. Another way in which an equity swap differs from an interest rate swap is that, while the floating rate on an interest rate swap is typically set at the beginning of the settlement period, the return on the stock is not known until the end of the period, at which time settlement is made.

In spite of these differences, equity swaps are remarkably similar to interest rate swaps. In fact as we shall see in this note, the rate on an equity swap where one party pays a fixed rate is the same as that on a plain vanilla interest rate swap.

The following material borrows heavily from my co-authored paper, "The Pricing of Equity Swaps and Swaptions," listed in the references.

In the examples here, we shall assume a \$1 notional principal. We shall also assume that the equity return is either from a non-dividend paying stock or index of stocks or that the equity

payoff is based on a total return index, which reflects capital gains and dividends. Specifically we define the equity factor as a total return index whose value at time j is $I(j)$. We let $B(j,k)$ represent the value at j of a zero coupon bond with \$1 face value that matures at k . Each swap we examine is a transaction in which the upcoming payments will be made at times $n+1, n+2, \dots, n+m$. The beginning of the current settlement period is at time n , with the previous payment made at time n . We are currently positioned at time j , $n \leq j < n+1$. When we price the swap at its initiation date, we let $j = n$ and call it time 0. We require no assumptions about the stochastic process associated with the index. Furthermore, we require no assumptions about the term structure other than it being observable and arbitrage-free. Hence, stochastic interest rates are permitted. When we price equity swaps with a variable notional principal, however, we shall require a deterministic term structure. Finally, we assume no taxes, transaction costs or market making. These factors can affect the pricing of equity swaps, but to include them here would not be beneficial.

Pricing Equity Swaps with a Fixed Notional Principal

In equity swaps with a fixed notional principal, one counterparty pays the return on the equity index applied to a constant notional principal. To the other counterparty this specification is equivalent to a transaction in which it has invested in the equity index but withdraws the return each period, a process that could be described as periodic rebalancing to a constant dollar investment. With one party paying the equity return, the other party can pay either a fixed return, a floating return or the return on another equity index. We examine each of these cases in detail.

To the party paying the fixed rate and receiving the index return, the stream of payments on this swap is as follows:

$$\text{At time } n+1: [I(n+1)/I(n)] - (1 + R)$$

$$\text{At time } n+2: [I(n+2)/I(n+1)] - (1 + R)$$

...

$$\text{At time } n+m: [I(n+m)/I(n+m-1)] - (1 + R)$$

This specification properly reflects the receipt of the percentage change in the index versus payment of a fixed rate per period of R .¹ At time j we define the value of a swap in which the current settlement period began at time n and the swap matures at time $n+m$ as $V(j;n,n+m)$, which is the present value of the above stream of payments.

We can replicate these payments as follows. Position ourselves at time j , $n < j < n+1$. To reproduce the first cash flow, simply invest $I(j)/I(n)$ dollars in the index and borrow $(1 + R)B(j,n+1)$. At time $n+1$, the cash flow will be $(I(j)/I(n))(I(n+1)/I(j)) - (1 + R) = I(n+1)/I(n) - (1 + R)$, which is the first cash flow on the swap. To reproduce the second cash flow, at time j borrow $(1 + R)B(j,n+2)$ and invest $B(j,n+1)$ in the risk-free asset. At $n+1$, the risk-free asset will be worth \$1, which should then be invested in the index. The debt is not due until $n+2$. At time $n+2$, we will have $I(n+2)/I(n+1)$, the value of our investment in the equity index, with debt due of $1 + R$. This reproduces the second cash payment of the swap. To reproduce each remaining cash payment, we follow a similar procedure. The value of these combined transactions at j is the same as the value of the swap as derived above, which is

$$V(j;n,n+m) = \frac{I(j)}{I(n)} - B(j,n+m) - R \sum_{i=1}^m B(j,n+i).$$

Our formula can be used to determine the equilibrium swap rate at the start of the swap by simply letting the start date of the swap be time j , which is set to time n , which we shall call time 0. Then setting (1) to zero and solving for the fixed rate gives

$$R = \frac{1 - B(0,m)}{\sum_{i=1}^m B(0,i)}.$$

Note that the level of interest rates at the start of the swap, and not the level of stock prices, completely determines the swap rate. During the life of the swap, however, Equation (1) shows that its value is determined by a combination of interest rate and stock price movements.

It may be surprising to learn that the fixed rate on an equity swap is the same as the fixed rate on a plain vanilla interest rate swap. To see this result, consider a plain vanilla swap where the present value of a series of floating payments less the present value of a series of fixed payments at the rate F is

¹In practice the rate R is usually stated as an annual rate so if the settlement period is not a year, then R is multiplied by (days/360) where days is the number of days in the settlement period. Sometimes 30 day months are assumed. Also 365 is sometimes used instead of 360.

$$1 - B(n, n + m) - F \sum_{i=1}^m B(n, n + i),$$

which, when set to zero and solved for F , gives the same formula as the one derived for R in Equation (2).

Some equity swaps are structured so that instead of paying a fixed rate, the party pays a floating rate such as LIBOR. Since we demonstrated that the fixed rate on an equity swap equals the fixed rate on a plain vanilla swap, it follows that the pay-floating, receive-equity swap is an equivalent exchange, meaning that it automatically has zero value at the start. It is not necessary to build in any terms to the contract. It is important, however, to obtain a general valuation formula that applies at any time during the life of this swap. A pay-floating, receive equity swap can be decomposed into a pay-fixed, receive-equity swap plus a plain vanilla pay-floating, receive-fixed interest rate swap. The value of the former is given by Equation (1). The value of the latter at time j , using the notation chosen here, is well known to be

$$R \sum_{i=1}^m B(j, n + i) + B(j, n + m) - [1 + r(n, n + 1)]B(j, n + 1),$$

where $r(n, n+1)$ is the floating rate set at n that will be paid at $n+1$. Adding this result to Equation (1) gives

$$V(j; n, n + m) = \frac{I(j)}{I(n)} - [1 + r(n, n + 1)]B(j, n + 1).$$

Of course at the start or immediately after a payment is made at any settlement date, the above expression will reduce to zero.²

Now consider a two-way equity swap where each party pays the equity return on a different stock or index. We assume that the returns are based on two domestic equity indices, I_1 and I_2 . We assume \$1 notional principals on both sides. From the perspective of the party paying the second index and receiving the first index, the first payment is

$$\frac{I_1(n+1)}{I_1(n)} - 1 - \left(\frac{I_2(n+1)}{I_2(n)} - 1 \right) = \frac{I_1(n+1)}{I_1(n)} - \frac{I_2(n+1)}{I_2(n)}.$$

Likewise, the second payment is

²This occurs because $I(n)/I(n) = 1$ and $[r(n, n+1) + 1]B(n, n+1) = 1$, the latter result due to the fact that it is equal to the value of a \$1 par floating rate bond at the upcoming coupon reset date.

$$\frac{I_1(n+2)}{I_1(n+1)} - \frac{I_2(n+2)}{I_2(n+1)}.$$

Thus, the last payment would be

$$\frac{I_1(n+m)}{I_1(n+m-1)} - \frac{I_2(n+m)}{I_2(n+m-1)}.$$

To price this swap we replicate its cash flows and set its value to the value of the replicating portfolio. To accomplish this, at time j we invest $I_1(j)/I_1(n)$ dollars in index 1 and sell short $I_2(j)/I_2(n)$ dollars of index 2. At time $n+1$ we liquidate these positions, which will produce a cash flow of $(I_1(j)/I_1(n))(I_1(n+1)/I_1(j)) - (I_2(j)/I_2(n))(I_2(n+1)/I_2(j)) = I_1(n+1)/I_1(n) - I_2(n+1)/I_2(n)$, which is the first cash flow of the swap. To reproduce the second cash flow, we need do nothing at time j . At time $n+1$ we sell short \$1 of index 2 and use the proceeds to buy \$1 of index 1. At time $n+2$ we liquidate the position for a cash flow of $I_1(n+2)/I_1(n+1) - I_2(n+2)/I_2(n+1)$, which is the second cash flow of the swap. Likewise to reproduce the k^{th} cash flow of the swap we need do nothing at time j , but at time $k-1$ we sell short \$1 of index 2 and use the proceeds to buy \$1 of index 1. Then at time k , we liquidate those positions. Consequently, at time j the value of the swap is the value of the initial long and short positions necessary to reproduce the first cash flow only, which is

$$V(j; n, n+m) = \frac{I_1(j)}{I_1(n)} - \frac{I_2(j)}{I_2(n)}.$$

All remaining cash flows can be reproduced with a self-financing strategy of shorting the second index and using the proceeds to buy the first index. At the start of the swap, we have time $0 = j = n$, which sets the value to zero. Obviously, no “pricing” is required as the series of futures cash flows is an equivalent exchange. Unlike the fixed-for-floating equity swap, however, two-way equity swaps have zero value immediately after payment at each reset date, as indicated by the fact that the remaining cash flows are replicated by shorting \$1 of index 2 and using the proceeds to buy \$1 of index 1.

Now consider a swap in which a party pays a domestic equity index return and receives a currency-adjusted return on a foreign index. Letting the domestic index be I_2 and the foreign index be I_1 , we obtain the valuation formula in a straightforward manner.

Let $S(n)$ be the spot exchange rate at time n and $S(n+1)$ be the spot exchange rate at time $n+1$, both defined as units of the domestic currency per unit of the foreign currency. The value of the next payment after conversion to the domestic currency will be

$$\frac{I_1(n+1)S(n+1)}{I_1(n)S(n)} - \frac{I_2(n+1)}{I_2(n)}.$$

Valuation of this type of swap is quite simple. We need only define the underlying index in terms of its value times the exchange rate $I_1(j)S(j)$. With this in mind, pricing the cross-currency two-way equity swap follows the same approach as the standard two-way equity swap. All swap payments beyond the first have zero value. Thus,

$$V(j; n, n+m) = \frac{I_1(j)S(j)}{I_1(n)S(n)} - \frac{I_2(j)}{I_2(n)}.$$

Again, we see that at initiation of the swap, time $0 = j = n$, the value is by definition zero, and no “pricing” is required. In fact, the standard two-way equity swap is just a special case of the cross-currency equity swap, the case in which both indices are denominated in the same currency and the exchange rate is always 1.0.

Pricing the Equity Swap with a Variable Notional Principal

Some equity swaps are structured with a variable notional principal. To see the reason for this suppose we entered into a two-year equity swap to receive the return on a stock index with annual payments on \$1,000 notional principal. The first year the stock went up by 10%, and the second year the stock went up by 15%. Under a fixed notional principal equity swap, we would receive \$100 the first year and \$150 the second year. Had we actually invested in the stock, however, the market value at the end of the first year would be \$1,100. Consequently, the second year our capital gain would be $\$1,100(0.15) = \165 . A variable notional principal would make the equity swap reproduce a transaction in which a given amount is initially invested in the index, returns are paid out and shares are purchased with the returns. Note, however, that adjusting the notional principal overcompensates, because we receive \$150 at the end of the first year and \$165 at the end of the second year. Had we bought the actual stock we would have earned \$165 at the end of the second year only if we had left the \$150 invested. In a sense this swap is leveraged, since it implicitly allows the party to receive the returns and borrow an equivalent amount to reinvest in the stock. Thus, this type of swap has greater upside potential and, consequently, must require a higher fixed payment. To offset this effect would require that

we make no payments until the end of the life of the swap and then pay out the return over the full life of the swap. Some swaps are structured this way, but they are easily accommodated by the first model we examined under the assumption of but one settlement period.

In this section we initialize the notional principal at \$1 and allow it to vary, as described above, according to the return on the underlying equity index.³ In this section we do only the standard pay-fixed, receive equity swap. The remaining swaps can be obtained from these results and those in the previous sections.

Letting the notional principal on payment $n+1$ be \$1, the first payment is

$$(\$1) \left[\frac{I(n+1)}{I(n)} - 1 - R \right].$$

The second payment is

$$\left(\frac{I(n+1)}{I(n)} \right) \left[\frac{I(n+2)}{I(n+1)} - 1 - R \right] = \frac{I(n+2)}{I(n)} - (1+R) \left[\frac{I(n+1)}{I(n)} \right].$$

Note how the first term in parentheses reflects the changing notional principal. The last payment is

$$\left(\frac{I(n+m-1)}{I(n)} \right) \left[\frac{I(n+m)}{I(n+m-1)} - 1 - R \right] = \frac{I(n+m)}{I(n)} - (1+R) \left[\frac{I(n+m-1)}{I(n)} \right].$$

At time j to reproduce the first payment we must invest $I(j)/I(n)$ dollars in the index and borrow $(1+R)B(j,n+1)$ dollars. Then at time $n+1$ we liquidate the index, which will create a cash flow of $(I(j)/I(n))I(n+1)/I(j) = I(n+1)/I(n)$, and pay off our loan, costing us $(1+R)$. The net cash flow is the same as that of the first swap payment.

Reproduction of the second cash flow is considerably more difficult. In fact indexation of the notional principal will force us to drop one of our highly valued assumptions, which is that interest rates are stochastic. We must now assume deterministic interest rates in order to know how much to borrow to reproduce the swap cash flow. At time j to reproduce the second cash flow we invest $I(j)/I(n)$ dollars in the index and sell short $(1+R)(I(j)/I(n))B(n+1,n+2)$ dollars of the index. Note that $B(n+1,n+2)$ is a forward discount bond price. At time $n+1$ our short position is worth $-(1+R)(I(j)/I(n))B(n+1,n+2)I(n+1)/I(j) = -(1+R)B(n+1,n+2)I(n+1)/I(n)$. We cover our short position by borrowing that amount and using it to buy back the stock. Then at

³In comparison to the example we use here, in some equity swaps the notional principal varies according to the net swap payment, which is the equity return minus the fixed rate. This case is easy to derive following the approach used here.
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time $n+2$ we liquidate the index position and receive cash of $(I(j)/I(n))I(n+2)/I(j) = I(n+2)/I(n)$. We then pay off the amount due on our loan, which is $-(1+R)B(n+1,n+2)(I(n+1)/I(n))/B(n+1,n+2) = -(1+R)I(n+1)/I(n)$. The overall cash flow matches the second cash flow on the swap.

To reproduce the k^{th} cash flow of the swap we follow the same procedure, investing $I(j)/I(n)$ dollars in the index and selling short $(1+R)(I(j)/I(n))B(k-1,k)$. At time $k-1$, we borrow $(1+R)B(k-1,k)I(k-1)/I(n)$ dollars and use the proceeds to cover our short position. Then at time k we liquidate both positions for a net cash flow of $I(k)/I(n) - (1+R)I(k-1)/I(n)$, which is the cash flow from the k^{th} swap payment.

The overall value of the swap at time j is the sum of the values of all of the transactions executed at time j , which is

$$V(j; n, n+m) = \frac{I(j)}{I(n)} \left[m - (1+R) \sum_{i=2}^m B(n+i-1, n+i) \right] - (1+R)B(j, n+1).$$

At the start of the swap, time $0 = j = n$, we simply set the above to zero and solve for R .

There are numerous other variations of equity swaps, including some that have caps and barriers. We refer the reader to Chance and Rich (1998) for material on the pricing of these exotic swaps.

The appendix contains numerical examples for each of the above types of swaps.

Appendix

Consider the following information, which defines the U. S. term structure and will be used to price and value equity swaps maturing in three periods. The notation $r(0,a)$ is the spot rate of interest for the period from time 0 to time a . $B(0,a)$ is the price of a zero coupon bond for the period 0 to a . The information used here is the same as that used in TN97-06: Pricing Interest Rate and Currency Swaps.

$$r(0,1) = 0.08, r(0,2) = 0.09, r(0,3) = 0.10$$

$$B(0,1) = 1/[1 + 0.08(360/360)] = 0.9259$$

$$B(0,2) = 1/[1 + 0.09(720/360)] = 0.8475$$

$$B(0,3) = 1/[1 + 0.10(1080/360)] = 0.7692$$

The forward prices will also be required. They are

$$B(1,2) = 0.8475/0.9259 = 0.9153$$

$$B(2,3) = 0.7692/0.8475 = 0.9076.$$

For purposes of analyzing swaps that involve stock indices of two countries, we shall require a second country, which we choose to be Switzerland. Let the spot exchange rate for Swiss francs be $\$S(0) = \0.70 . The total return index representing the U. S. stock market, $I_2(0)$ is worth 1,200.00, and the index representing the Swiss market is worth $I_1(0) = 7,700.00$.

For a receive-equity, pay-fixed swap based on the U. S. total return index with a fixed notional principal, the fixed rate is easily found from Equation (2):

$$R = \frac{1 - 0.7692}{0.9259 + 0.8475 + 0.7692} = 0.0908.$$

Recall that this is the same rate as that on the plain vanilla interest rate swap, which we found in the appendix to TN97-06.

For a receive-equity, pay-floating swap, both sides are variable, and there is no need to derive pricing terms because the swap automatically has zero present value at the start.

Suppose that the swap involves the payment of the total return on another U. S. stock index, which we shall assume is at 580.00. As in the receive-equity, pay-floating swap, there is no need to solve for pricing terms at the start, since both streams of cash flows have equivalent value.

Now consider the swap in which we agree to pay based on the U. S. total return index and receive payment based on the Swiss total return index. Once again, with both sides of these payments being variable, there is no need to solve for pricing terms at the start.

Now let us go back to the receive-equity, pay-fixed swap but make the notional principal be variable. At the start of the swap, using Equation (6), we have

$$V(0;0,3) = \frac{1,200.00}{1,200.00} [3 - (1+R)(0.9153 + 0.9076)] - (1+R)0.9259 = 0.0$$

Solving for R gives 0.0901.

Valuation of Equity Swaps

Now let us move six months into the swap's life and determine the values of these equity swaps given that we now have new information. We have a new term structure in the U. S. and Switzerland as shown below:

$$r(0.5,1) = 0.082, r(0.5,2) = 0.094, r(0.5,3) = 0.105$$

$$B(0.5,1) = 1/[1 + 0.082(180/360)] = 0.9606$$

$$B(0.5,2) = 1/[1 + 0.094(540/360)] = 0.8764$$

$$B(0.5,3) = 1/[1 + 0.105(900/360)] = 0.7921$$

The forward prices are

$$B(1,2) = 0.8764/0.9606 = 0.9123$$

$$B(2,3) = 0.7921/0.8764 = 0.9038$$

The forward rates are

$$r(0.5,1,2) = \{([1 + 0.094(540/360)]/[1 + 0.082(180/360)]) - 1\}(360/360) = 0.0961$$

$$r(0.5,2,3) = \{([1 + 0.105(900/360)]/[1 + 0.094(540/360)]) - 1\}(360/360) = 0.1065$$

These rates imply forward prices of $1/1.0961 = 0.9123$ and $1/1.1065 = 0.9038$, which we shall need.

Let the new exchange rate be \$0.725. The U. S. stock index has now gone to 1207.25 and the Swiss index has gone to 7810.50. The other U. S. stock index is now 591.15.

The receive-equity, pay-fixed swap with fixed notional principal has a value given by Equation (1) of

$$V(0.5;0,3) = \frac{1,207.25}{1,200.00} - 0.7921 - 0.0908(0.9606 + 0.8764 + 0.7921) = -0.0248.$$

The receive-equity, pay-floating swap with fixed notional principal has a value given by Equation (3) of

$$V(0.5;0,3) = \frac{1,207.25}{1,200.00} - (1.08)(0.9606) = -0.0314.$$

The relationship between these two equity swaps and a plain vanilla swap is easily seen in these values. If we also entered into the first swap (receiving equity and paying fixed), we would have a value of -0.0248. If also we entered into the opposite version of the second swap (receiving floating and paying equity), it would have a value of 0.0314. We would then have the equity payments cancel, leaving us with receiving floating and paying fixed, a plain vanilla swap. The difference in these two swap values, $-0.0248 + 0.0314 = 0.0066$, is the value of a plain vanilla swap, which we determined in the appendix to TN97-06.

Now consider the two-way equity swap in which we are paying the return on the index originally at 1200.00 and receiving the return on the index originally at 580.00. With the two indices now at 591.15 and 1,207.25, the value of the swap from Equation (4) is

$$V(0.5;0,3) = \frac{591.15}{580.00} - \frac{1,207.25}{1,200.00} = 0.0132.$$

Now consider the swap in which we pay the return on the U. S. index and receive the return in Swiss francs on the Swiss index. From Equation (5) this swap has a value of

$$V(0.5;1,3) = \frac{7,810.50(0.725)}{7,700.00(0.70)} - \frac{1,207.25}{1,200.00} = 0.0445.$$

Note how value changes arise from changes in both the stock indices and the exchange rate.

Recall that when pricing the swap with variable notional principal, we were forced to assume deterministic interest rates. Now we have allowed interest rates to change along with the stock index. We can find the new value based on the formula, which will be

$$V(0.5;0,3) = \frac{1,207.25}{1,200.00} \left[3 - (1.0901)(0.9123 + 0.9038) \right] - (1.0901)(0.9606) = -0.0207.$$

There would also appear to be an inconsistency in pricing the swap this way. As we have seen, the no-arbitrage argument does not hold if interest rates are stochastic. Note, however, that this type of problem is the same as assuming constant interest rates and volatility in the Black-Scholes model or constant interest rates and yield in the cost of carry model for forwards. It is not the ideal way to do it, but it is probably better than a more complex model.

References

There is very little literature on equity swaps. Listed below are the primary sources.

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