TEACHING NOTE 98-02: APPROXIMATION OF AMERICAN OPTION VALUES: BARONE-ADESI-WHALEY

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An approximation formula for American calls and puts on both assets and futures was developed by Barone-Adesi and Whaley (1987). Using a technique originally developed by Macmillan (1986), they split the option price into two components: the European option value and the early exercise premium. They then obtain the partial differential equation for the early exercise premium, make some simplifying assumptions and obtain a quadratic equation that approximates the solution.

We begin by allowing a yield on the asset of y. Then define the asset's cost of carry to be b = r - y. It is well-known that the partial differential equation that describes the option's price is given as

$$\frac{1}{2}\sigma^2 S^2 C_{SS} + bSC_S - rC + C_t = 0$$

where C is the American option price, t is the present time and the subscripts indicate partial derivatives. When y = 0, b = r and this is the Black-Scholes PDE. When r = y, b = 0 and this is the Black option on futures pricing model. This result is convenient for it will allow us to simply set b to the value of r and use our solution to obtain the value of an American option on a futures.

We define the early exercise premium as

$$\varepsilon_C(S,T) = C(S,T) - c(S,T)$$

where S, the stock price, and T, the time to expiration, are the arguments that can change. The PDE for the early exercise premium is

$$\frac{1}{2}\sigma^2 S^2 \varepsilon_{SS} + bS\varepsilon_S - r\varepsilon + \varepsilon_t = 0.$$

Letting t* be the option's expiration and define T = t* - t to be the time to expiration. Therefore, $\varepsilon_T = -\varepsilon_t$. Now we multiply the above equation by $2/\sigma^2$ and define $M = 2r/\sigma^2$ and $N = 2b/\sigma^2$. Then we have

$$S^2\varepsilon_{SS} - M_\varepsilon + NS\varepsilon_S - (M/r)\varepsilon_T = 0.$$

Now define $\varepsilon_C(S,K) = K(T)f(S,K)$, which is just a very general specification for the early exercise premium as a function of the stock price and time to expiration. Then $\varepsilon_{SS} = Kf_{SS}$ and $\varepsilon_T = Ktf + KK_Tf_K$. Substitute these results into the above equation, factor terms and gather common terms on Mf gives us

$$S^2f_{SS} + NSf_S - M_f[\ 1 + (K_T/rK)(1 + Kf_K/f)] = 0.$$

Now choose $K(T) = 1 - e^{-rT}$ and substitute into the above, giving us

$$S^2f_{SS} + NSf_S - (M/K)f - (1 - K)Mf_K = 0.$$

So far no approximation has been made. Then they let $(1 - K)Mf_K = 0$. With a short (long) value of T, $f_K \rightarrow 0$ $(K \rightarrow 1)$. This leaves

$$S^2f_{SS} + NSf_S - (M/K)f = 0.$$

This is a second order ordinary differential equation with two linearly independent solutions of the form αS^q. Let $f = aS^q$ and subsitute into the above giving

$$aS^q[q^2 + (N-1)q - M/K] = 0.$$

The roots of this equation are

$$q_1 = \left[-\left(N-1-\sqrt{(N-1)^2+4M/K}\right)\right]\Big/2$$

$$q_2 = \left[-\left(N-1+\sqrt{(N-1)^2+4M/K}\right)\right]\Big/2.$$

Because $M/K > 0$, $q_1 < 0$ and $q_2 > 0$. The general solution is

$$f(S) = a_1S_1^q + a_2S_2^q.$$

We know q_1 and q_2 and, thus, need to know a_1 and a_2.

With $q_1 < 0$, $a_1 \neq 0$, $f \rightarrow \infty$ and $S \rightarrow 0$. This is not an acceptable result, because as $S \rightarrow 0$, ε_C should approach zero. Let $a_1 = 0$. Then we have

$$C(S,T) = c(S,T) + Ka_2S_2^q.$$

As $S \rightarrow 0$, $C(S,T) \rightarrow 0$. As S increases, C(S,T) should approach S - X. The equation,

$$S* - X = c(S*,T) + Ka_2S^{*q}_2,$$

with S* the critical stock price to trigger early exercise, must hold. Also the slopes must be equal, so differentiating the above equation, we obtain

$$1 = e^{(b-r)T}N(d_1(S*) + Kq_2a_2S^{*q-1}_2.$$

So we solve these two equations for a_2 and S^*. For a_2 we have

$$a_2 = \{1 - e^{(b - r)T}N[d_1(S*)]\}/Kq_2 S^{*q-1}_2 .$$

We then substitute this result into the other equation to obtain

$$S^* - X = c(S^*,T) + \{1 - e^{(b - r)T}N[d_1(S*)]\}S*/q^2$$

This gives us the following:

$$C(S,T) = c(S,T) + A_2(S/S^*)^q{}_2 \text{ when } S < S*$$

$$C(S,T) = S - X \text{ when } S \geq S*$$

and

$$A_2 = (S^*/q_2)\{1 - e^{(b - r)T}N[d_1(S*)]\},$$

which will be greater than zero when $b < r$. When $b \geq r$, implying $y < 0$, the option will never be exercised early so the Black-Scholes formula holds.

To price the American put we follow the same basic procedure. The value of ε_P must approach zero as $S \to \infty$. The term $a_2S^q{}_2$ violates this condition. We set a_2 to zero and obtain

$$P(S,T) = p(S,T) + Ka_1S^q{}_1.$$

Following a similar approach as we did when the option was a call, we have

$$a_1 = -\{1 - e^{(b - r)T}N[-d_1(S*)]\}/Kq_1 S^{**q-1}_1$$

where S^{**} is the critical price for exercising the put early. We have $a_1 > 0$ since $q_1 < 0$. The critical stock price is defined as

$$X - S^{**} = p(S*,T) - \{1 - e^{(b - r)T}N[-d_1(S*)]\}S*/q_1.$$

Thus, we have

$$P(S,T) = p(S,T) + A_1(S/S*)^q{}_1 \text{ when } S > S**$$

$$P(S,T) = X - S \text{ when } S \leq S*$$

and

$$A_1 = (S*/q_1)\{1 - e^{(b - r)T}N[-d_1(S*)]\},$$

which is positive.

For an option on a futures contract, we simply set b to r and use the above formulas with the futures price instead of the asset price. This is consistent with the fact that b is the cost of carry and the cost of carry on a futures contract is zero. Thus an option on a futures is an option on an instrument that has a zero cost of carry. This rule applies to all types of options and options on futures.

To implement the model it is necessary to obtain the critical stock prices S^* and S^{**}. These can be found by an iterative search procedure with a starting value of the exercise price or using a more efficient search procedure such as Newton-Raphson, which is demonstrated in the Barone-Adesi-Whaley article. They also present a method for obtaining a value at which to start the search for S^* and S^{**}.

Empirical tests of the model for options on futures are found in Whaley (1987).

References

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Some of the early work on approximation formulas for American options is

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