

TEACHING NOTE 98-04: EXCHANGE OPTION PRICING

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The exchange option, first developed by Margrabe (1978), has proven to be an extremely powerful generalization of the Black-Scholes model. The intuitive insights in the derivation of the exchange option model are very useful in other applications in option pricing.

Let us consider a European option in which at expiration, the holder can exchange one unit of asset 2 and receive one unit of asset 1. Let $c_t(S_1, S_2)$ denote the price of this call option. Its payoff at expiration is $c_T(S_1, S_2) = \text{Max}(0, S_1 - S_2)$. In this option, asset 2 plays the role of the exercise price, but asset 2 is stochastic. Alternatively, one can view this option as a put in which asset 2 can be exchanged for asset 1 if asset 2 has lower value. Let this option's price be denoted as $p_t(S_2, S_1)$ and payoff at expiration be $p_T(S_2, S_1) = \text{Max}(0, S_1 - S_2)$. We first establish some boundary conditions on the price of this option.

Establishing a Lower Bound and Put-Call Parity

We begin by constructing a lower bound, which will also show the relationship between the exchange option price and the spread between the prices of the two underlying assets. At time t , construct portfolio A by purchasing one exchange option. Then construct portfolio B by purchasing one unit of asset 1 and selling short one unit of asset 2. The payoffs are shown below.

			Value at Expiration	
	Instrument	Current Value	$S_{1T} \leq S_{2T}$	$S_{1T} > S_{2T}$
Portfolio A	Exchange option	$c_t(S_1, S_2)$	0	$S_{1T} - S_{2T}$
Portfolio B	Long asset 1	S_{1t}	S_{1T}	S_{1T}
	Short asset 2	$-S_{2t}$	$-S_{2T}$	$-S_{2T}$
	Total	$S_{1t} - S_{2t}$	$S_{1T} - S_{2T}$	$S_{1T} - S_{2T}$

It is clear that portfolio A dominates portfolio B so the value of A at time t must be no less than the value of B at time t. Since the exchange call cannot be worth less than zero, we can say that its minimum value is given as

$$c_t(S_1, S_2) \geq \text{Max}(0, S_{1t} - S_{2t}).$$

Note that if the exchange call were an American option, early exercise would generate only $S_{1t} - S_{2t}$ so it would never be more valuable to exercise it early. Thus, the American exchange call would be worth the same as the European exchange call.

Now we develop a type of put-call parity. Let Portfolio A consist of the same exchange call and a short put to exchange asset 2 for asset 1. Let Portfolio B be the same as above, a long position in asset 1 and a short position in asset 2.

			Value at Expiration	
	Instrument	Current Value	$S_{1T} \leq S_{2T}$	$S_{1T} > S_{2T}$
Portfolio A	Long exch. call to swap asset 2 for asset 1	$c_t(S_1, S_2)$	0	$S_{1T} - S_{2T}$
	Short exch. put to swap asset 1 for asset 2	$p_t(S_1, S_2)$	$-(S_{2T} - S_{1T})$	0
	Total	$c_t(S_1, S_2) - p_t(S_1, S_2)$	$S_{1T} - S_{2T}$	$S_{1T} - S_{2T}$
Portfolio B	Long asset 1	S_{1t}	S_{1T}	S_{1T}
	Short asset 2	$-S_{2t}$	$-S_{2T}$	$-S_{2T}$
	Total	$S_{1t} - S_{2t}$	$S_{1T} - S_{2T}$	$S_{1T} - S_{2T}$

Here portfolios A and B produce equivalent payoffs; thus, their initial values must be the same. Consequently,

$$c_t(S_1, S_2) - p_t(S_1, S_2) = S_{1t} - S_{2t}.$$

The same statement also holds for American exchange options. Note the similarity between put-call parity for exchange options and put-call parity for ordinary options.¹ If we replaced S_{2t} with $Xe^{-r(T-t)}$

¹Note that put-call parity can also be written as $c_t(S_1, S_2) - c_t(S_2, S_1) = S_{1t} - S_{2t}$.

in the above formula, we would have put-call parity for ordinary options. As we shall soon see, there is a most intuitive reason for this substitution.

Pricing the Exchange Option

Before beginning our derivation of the exchange option pricing model, let us review the mathematical principle of homogeneity, which plays an important role. Consider an unspecified mathematical function $f(x_1, x_2, \dots, x_n)$. The function is said to be homogeneous of degree λ with respect to every variable x if $f(ax_1, ax_2, \dots, ax_n) = a^\lambda f(x_1, x_2, \dots, x_n)$ for constant a . For example, consider the function $g(x, y, z) = 2x^2 + 3yz - z^2$, which is homogeneous of degree two with respect to every variable. (Note that $2a^2x^2 + 3ayz - a^2z^2 = a^2g(x, y, z)$.) A function is homogeneous of degree zero with respect to every variable if its value is not altered when multiplying each term by some constant, a . A function can be homogeneous with respect to a limited number of variables. For example, suppose $f(ax_1, x_2, \dots, x_n) = a^\lambda f(x_1, x_2, \dots, x_n)$. Then $f(x_1, x_2, \dots, x_n)$ is said to be homogeneous of degree λ with respect to x_1 . A function that is homogeneous of degree one is said to be *linearly homogeneous*. Suppose $f(x_1, x_2, \dots, x_n)$ is linearly homogeneous with respect to x_1 and x_2 . This means that $f(ax_1, ax_2, x_3, \dots, x_n) = a f(x_1, x_2, \dots, x_n)$. Euler's Theorem permits us to state that $f(x_1, x_2, \dots, x_n) = x_1 \partial f(\bullet) / \partial x_1 + x_2 \partial f(\bullet) / \partial x_2$. This result will prove useful in deriving the exchange option pricing model.

Now let us propose that there are two assets, $i = 1, 2$, each following its own lognormal diffusion,

$$\frac{dS_{it}}{S_{it}} = \alpha_i dt + \sigma_i dz_{it}$$

and the correlation between the two Weiner processes driving the asset prices is ρ_{12} . Let $c_t(S_{1t}, S_{2t})$ be the value at t of the exchange option, which gives the right to tender asset 2 for asset 1 at expiration, T . As noted above, the payoff of this option is $c_T(S_1, S_2) = \text{Max}(0, S_1 - S_2)$. It is easy to see that this terminal payoff is linearly homogeneous with respect to the two asset values. Since the value of the option today is a simple discounted expectation of its payoff at expiration, its current value must also be linearly homogeneous.²

²Taking the discounted expectation is a linear operation, which preserves the linear homogeneity.

For notational simplicity, let us temporarily denote the option's value as c_t . Using Euler's Theorem, we can express the value of the option as

$$c_t - \frac{\partial c_t}{\partial S_{1t}} S_{1t} - \frac{\partial c_t}{\partial S_{2t}} S_{2t} = 0.$$

This statement has a natural interpretation: a portfolio consisting of the purchase of one unit of the exchange call and short positions in $\partial c_t / \partial S_{1t}$ units of asset 1 and $\partial c_t / \partial S_{2t}$ units of asset 2 would require no initial investment. To avoid profitable arbitrage, such a portfolio must generate an instantaneous return of zero:

Now we apply the multivariate version of Itô's Lemma to the value of the call:

$$dc_t = \frac{\partial c_t}{\partial S_{1t}} dS_{1t} + \frac{\partial c_t}{\partial S_{2t}} dS_{2t} + \frac{\partial c_t}{\partial t} dt + \frac{1}{2} \left(\frac{\partial^2 c_t}{\partial S_{1t}^2} \sigma_1^2 S_{1t}^2 + 2 \frac{\partial^2 c_t}{\partial S_{1t} \partial S_{2t}} \sigma_1 \sigma_2 \rho_{12} S_{1t} S_{2t} + \frac{\partial^2 c_t}{\partial S_{2t}^2} \sigma_2^2 S_{2t}^2 \right) dt$$

Thus, with $c_t - (\partial c_t / \partial S_{1t}) S_{1t} - (\partial c_t / \partial S_{2t}) S_{2t} = dc_t - (\partial c_t / \partial S_{1t}) dS_{1t} - (\partial c_t / \partial S_{2t}) dS_{2t} = 0$, substituting Itô's Lemma for dc_t , we obtain the partial differential equation,

$$\frac{\partial c_t}{\partial t} + \frac{1}{2} \left[\frac{\partial^2 c_t}{\partial S_{1t}^2} \sigma_1^2 S_{1t}^2 + 2 \frac{\partial^2 c_t}{\partial S_{1t} \partial S_{2t}} \sigma_1 \sigma_2 \rho_{12} S_{1t} S_{2t} + \frac{\partial^2 c_t}{\partial S_{2t}^2} \sigma_2^2 S_{2t}^2 \right] = 0$$

The solution is

$$c_t(S_{1t}, S_{2t}) = S_{1t} N(d_1) - S_{2t} N(d_2)$$

where

$$d_1 = \frac{\ln(S_{1t}/S_{2t}) + (\sigma^2/2)(T-t)}{\sigma\sqrt{T-t}}, \quad d_2 = d_1 - \sigma\sqrt{T-t}$$

$$\sigma = \sqrt{\sigma_1^2 + \sigma_2^2 - 2\rho_{12}\sigma_1\sigma_2}.$$

Note that σ is the volatility of a new variable, the proportional change in the ratio S_{2t}/S_{1t} .

We can check our solution by taking the derivatives but a simpler and more intuitive check is possible. Since c_t is linearly homogeneous with respect to the prices of assets 1 and 2, we can say that $ac_t = c_t(aS_1, aS_2)$ where a is a constant. Define a as $1/S_2$, which gives us $(1/S_2)c_t(S_1, S_2) = c_t(S, 1)$ or $c_t(S_1, S_2) = S_2c(S, 1)$ where $S = S_1/S_2$. In effect we have created a new somewhat artificial asset, the ratio of the value of asset 1 to the value of asset 2. In other words, the exchange call can be expressed as S_2 units of an asset that allows the exchange of one dollar for the value of S_1 over S_2 . The latter is an ordinary European call on S with an exercise price of 1. We can, therefore, differentiate the exchange option price by differentiating its equivalent value, $S_2c(S, 1)$:

$$\frac{\partial c_t(S_1, S_2)}{\partial S_1} = \frac{\partial c_t(S_1, S_2)}{\partial S}, \quad \frac{\partial^2 c_t(S_1, S_2)}{\partial S_1^2} = \frac{\partial c_t(S_1, S_2)}{\partial S^2} \frac{S}{S_1}, \quad \frac{\partial c_t(S_1, S_2)}{\partial t} = S_2 \frac{\partial c_t(S_1, S_2)}{\partial t},$$

$$\frac{\partial c_t(S_1, S_2)}{\partial S_2} = \frac{\partial c_t(S_1, S_2)}{\partial S} - S \frac{\partial c_t(S_1, S_2)}{\partial S}, \quad \frac{\partial^2 c_t(S_1, S_2)}{\partial S_2^2} = \frac{\partial^2 c_t(S_1, S_2)}{\partial S^2} \frac{S^2}{S_2},$$

$$\frac{\partial^2 c_t(S_1, S_2)}{\partial S_1 \partial S_2} = - \frac{\partial^2 c_t(S_1, S_2)}{\partial S^2} \frac{S^2}{S_1}.$$

Making the appropriate substitutions and dividing by S_2 , we obtain the new partial differential equation,

$$\frac{\partial c_t}{\partial t} + \frac{1}{2} \frac{\partial^2 c_t}{\partial S^2} S^2 (\sigma_1^2 + \sigma_2^2 - 2\sigma_1\sigma_2\rho_{12}) = 0.$$

This partial differential equation is the same as the Black-Scholes partial differential equation when the interest rate, r , is set to zero, the underlying asset price is S , and the volatility is σ . Consequently, we can say that the exchange option is equivalent to S_2 units of an ordinary European call when the underlying asset is S , the strike is one, the interest rate is zero, and the volatility is $\sigma = \sigma_1^2 + \sigma_2^2 - 2\rho_{12}\sigma_1\sigma_2$.

This result is useful in better understanding not only the exchange option, but also the ordinary European option. The latter can be viewed as an exchange option where the asset exchanged is cash. The exchange option implies a zero interest rate because it can be replicated by holding asset 1 and shorting asset 2. The shorting of asset 2 would not have an expected return of r , as it would if it were risk-free. Rather the holder of asset 2 would demand its expected return, α_2 , as compensation. Consequently, the short seller of asset 2, who is trying to replicate the exchange option, would not have an expected return of $-r$ but rather of $-\alpha_2$. In any ordinary European call, the second term in the pricing equation is the present value of the exercise price. In the exchange option, the second term is also the present value of the exercise price. The current price of asset 2 is its present value.

Interestingly, in the same issue of *The Journal of Finance* directly preceding the Margrabe article, there is an article by Stanley Fischer (1978), in which he modeled bonds indexed to inflation. He showed that to price such a bond one needs the formula for an option where the exercise price is stochastic. Such an option is equivalent to an exchange option, and naturally Fisher derives the same formula as Margrabe.

References

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