

**TEACHING NOTE 98-05:
COMPOUND OPTION PRICING**

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A compound option is an option on an option. In other words, the underlying is another option. The original development of the compound option model was Geske (1979). The context in which this model was developed was that of a call option on a stock, which is itself a call option on the assets of the firm. Although the Black-Scholes model would appear to properly price the call option, if all the basic assumptions are met, Geske's compound option model shows that the volatility of the stock will be changing, owing to its status as an option on the assets.

Define the following terms:

- V_t = value of the firm's assets at time t
- M = face value of zero coupon debt issued by the firm and maturing at T
- S_t = market value of the stock of the firm at time t
- σ_v = volatility of the log return of the assets of the firm

Again, r is the risk-free rate and $T - t$ is the maturity of the debt. The equity is an option on the assets with payoff at T of $\text{Max}(0, V_T - M)$. With the assets following the lognormal diffusion, $dV/V = \alpha dt + \sigma dz$ and the usual Black-Scholes assumptions, the equity can be valued as

$$S_t = V_t N(d_1) - Me^{-r(T-t)} N(d_2)$$
$$d_1 = \frac{\ln(V_t/M) + (r + \sigma_v^2/2)(T-t)}{\sigma_v \sqrt{T-t}}$$
$$d_2 = d_1 - \sigma_v \sqrt{T-t}.$$

Now suppose there is a call option on the stock expiring at τ with an exercise price of X . Let us now derive the value of this call option in terms of the underlying asset, V , the value of the firm.¹ We construct a hedge portfolio by purchasing n_1 units of the asset and n_2 call options. The value of this portfolio is initially $H = n_1 V + n_2 c$.² We know that the change in the value of

¹We have eliminated the subscript where possible to reduce the notational complexity.

²At this point we diverge slightly from the original Geske derivation.

this portfolio is given by the total differential, $dH = (\partial H/\partial V)dV + (\partial H/\partial c)dc$. Since $H = n_1V + n_2c$, we know that $\partial H/\partial V = n_1$ and $\partial H/\partial c = n_2$. Thus,

$$dH = n_1dV + n_2dc.$$

Since the price of the call is a function of V and t , Itô's Lemma permits us to express the change in the call price as

$$dc = \frac{\partial c}{\partial t}dt + \frac{\partial c}{\partial V}dV + \frac{1}{2} \frac{\partial^2 c}{\partial V^2} V^2 \sigma_v^2 dt.$$

Substitute this result into the right-hand side of the above expression for dH . This gives

$$dH = n_1dV + n_2 \frac{\partial c}{\partial t}dt + n_2 \frac{\partial c}{\partial V}dV + n_2 \frac{1}{2} \frac{\partial^2 c}{\partial V^2} V^2 \sigma_v^2 dt.$$

Since we are free to set n_1 and n_2 , then let $n_1 = -n_2(\partial c/\partial V)$. Substituting into the above expression for dH gives

$$dH = n_2 \frac{\partial c}{\partial t}dt + n_2 \frac{1}{2} \frac{\partial^2 c}{\partial V^2} V^2 \sigma_v^2 dt.$$

This expression has no stochastic terms so it is risk-free. Therefore, the value of the hedge portfolio, H , should grow at the risk-free rate. Thus, we specify that $dH = Hr dt$. Substituting $(n_1V + n_2c)$ for H , $-n_2(\partial c/\partial V)$ for n_1 , and the above expression for dH , we obtain

$$n_2 \frac{\partial c}{\partial t}dt + n_2 \frac{1}{2} \frac{\partial^2 c}{\partial V^2} V^2 \sigma_v^2 dt = -n_2 \frac{\partial c}{\partial V} Vr dt + n_2 cr dt.$$

Dividing by n_2 and dt gives

$$cr = \frac{\partial c}{\partial V} Vr + \frac{\partial c}{\partial t} + \frac{1}{2} \frac{\partial^2 c}{\partial V^2} V^2 \sigma_v^2,$$

which appears to be the same partial differential equation we obtained for the standard European option.³ Unfortunately, there are some complicating factors. The solution for the call price in terms of the asset price is made difficult by the fact that the call will exercise or not at an intervening time based on the value of the equity relative to the call's strike, X . The call is not simply worth $V_T - X$ or zero at T . The call payoff is made at time τ and is expressed as $c_\tau = \text{Max}(0, S_\tau - X)$ but S_τ itself solves another partial differential equation because it is a standard

³Geske's alternative formulation requires that we express the stochastic process for the call as $dc/c = \alpha_c dt + \sigma_c dz_c$, which will imply that the market price of risk for the call, $(\alpha_c - r)/\sigma_c$, equals the market price of risk of the asset, $(\alpha_v - r)/\sigma_v$. Although this statement will be true under the assumptions of the model, introducing it without proof and relying on it can cause some confusion.

European option expiring later at T. Geske utilizes the principle of risk neutral valuation to price the option. Specifically, he evaluates the expression $c_t = e^{-r(\tau-t)}E[\text{Max}(0, S_\tau(V_\tau) - X)]$. This can be expressed as ⁴

$$c_t = e^{-r(\tau-t)} \int_{-\infty}^{\infty} [\text{Max}(0, s_\tau(Ve^u) - X)] f(u) du$$

where

$$u = \ln(S_\tau/S_t), \quad f(u) = \frac{e^{-(1/2)q^2}}{\sigma_v \sqrt{2\pi(\tau-t)}}$$

$$q = \frac{u - \mu(\tau-t)}{\sigma_v \sqrt{(\tau-t)}}, \quad \mu = r - \sigma_v^2/2.$$

These expressions simply arise from the fact that V_τ is lognormally distributed and the expected return is set to the risk-free rate. The problem can be broken down into three parts that have logical interpretations. If the option expires in the money at τ , the holder will exercise it and obtain a position in the stock. When the bonds mature at T, the stock value behaves like an ordinary call with a payoff equal to the expected value of the assets conditional on the bonds being paid off minus the expected payoff on the bonds. Thus, the value of the call can be expressed as (1) the discounted expected value of the assets at T when the bond matures conditional upon the call expiring in the money at τ minus (b) the discounted expected payout on the bond minus (c) the discounted expected payout of the exercise price on the option. These can be expressed in integral form as

$$(a) \quad Ve^{-r(T-t)} \int_{\ln(V^*/V)}^{\infty} e^u N(z) f(u) du$$

$$(b) \quad Me^{-r(\tau-t)} \int_{\ln(V^*/V)}^{\infty} N(z - \sigma_v \sqrt{T-\tau}) f(u) du$$

$$(c) \quad Xe^{-r(\tau-t)} \int_{\ln(V^*/V)}^{\infty} f(u) du,$$

where $z = [(\ln(V^*/M) + (r + \sigma_v^2/2)(T-\tau))] / \sigma_v \sqrt{T-\tau}$. The term V^* is the critical value of the assets at τ at which the equity would be sufficiently valuable to have the call option expire in-the-money. It can be found by solving the following equation for V^* :

⁴ At this point, we begin borrowing from Rubinstein (1991-92), who shows more of the details of the solution.

where $N_1(d_i)$, $i = 1,2$ is the standard normal probability distribution. The intuition behind this equation is easy to see. At time τ , the call expires and is exercisable if the stock price exceeds the exercise price. The left-hand side, $S_\tau(V^*) - X$, is the exercise value of the call. It is set equal to the right-hand side, which is the value of the stock minus the exercise price. The value of the stock is priced as a Black-Scholes option, with an exercise price of M , a volatility of σ_v and a time to expiration of $T - \tau$. The critical value of the assets, V^* , appears on both the left- and right-hand sides so the solution must be found iteratively.

The three equations (a), (b), and (c) have known solutions given as follows:

$$\begin{aligned} \text{(a)} \quad & VN_2(x, y; \rho) \\ \text{(b)} \quad & Me^{-r(T-t)}N_2(x - \sigma_v\sqrt{\tau-t}, y - \sigma_v\sqrt{T-t}; \rho) \\ \text{(c)} \quad & Xe^{-r(\tau-t)}N_1(x - \sigma_v\sqrt{\tau-t}) \end{aligned}$$

Where

$$\begin{aligned} x &= \frac{\ln(V/V^*) + (r + \sigma_v^2/2)(\tau - t)}{\sigma_v\sqrt{\tau - t}} \\ y &= \frac{\ln(V/M) + (r + \sigma_v^2/2)(T - t)}{\sigma_v\sqrt{T - t}} \\ \rho &= \sqrt{(\tau - t)/(T - t)}. \end{aligned}$$

The overall price of the compound option is (a) - (b) - (c).

Geske also provides the derivatives of the compound call price with respect to the underlying variables V , M , $T - t$, r , σ_v , X and $\tau - t$.⁵ These provide some interesting results. For example, $\partial c/\partial M < 0$, meaning that increasing the firm's leverage, which raises the variance of the equity, then increases the value of the call; however, the larger debt value combined with a fixed value of the assets, lowers the value of the equity, which is the dominant effect. This lowers the call price. Another interesting result is that the volatility of the stock is not constant. Define $V = B + S$. Then the total differential of V is $dV = (\partial V/\partial B)dB + (\partial V/\partial S)dS$. Letting dB equal zero, we have $dV = (\partial V/\partial S)dS$. Then $dV/V = (\partial V/\partial S)(dS/V)/(S/S)$. This can be written as $dV/V = \partial V/\partial S(S/V)(dS/S)$. The standard deviation of the asset return, $\sigma(dV/V)$ is, therefore, $(\partial V/\partial S)(S/V)\sigma(dS/S)$. Consequently, $\sigma_v = (\partial V/\partial S)(S/V)\sigma_s$. Turning this around, we have $\sigma_s =$

⁵ A useful technique to taking these derivatives is to use the derivatives in the Black-Scholes model. Hence, $\partial c/\partial w = (\partial c/\partial S)(\partial S/\partial w)$ where w is the variable of differentiation and $\partial S/\partial w$ is obtained from the standard Black-Scholes derivatives.

$\sigma_v(\partial S/\partial V)(V/S)$. The stock volatility is, thus, seen as a function of the asset volatility and the firm's leverage, which is picked up in the elasticity factor, $(\partial S/\partial V)(V/S)$.⁶

The significance of this result is that the volatility of the stock, which is what we would normally insert into the Black-Scholes model to obtain the call price, is definitely not constant as the Black-Scholes model assumes. The volatility of the assets may well be constant, but the firm's leverage changes with any change in the market value of the equity relative to the market value of the assets.

Geske goes on to show that the compound option is linearly homogeneous with respect to the value of the firm, the face value of the debt and the exercise price and that the compound option is convex in the value of the firm. He also shows that $M = 0$ will cause the model to converge to the Black-Scholes formula. This will also occur if the bond is a perpetuity or if the option expiration coincides with the bond maturity. In the latter case the two strikes merge to $M + X$.

Finally we should note that Rubinstein has generalized the Geske result to accommodate the other possible compound options: a call on a put, a put on a call and a put on a put as well as the case of a continuous payout on the assets of δ . The general formula appears below with $\eta = 1$ if the underlying option is a call and -1 if a put and $\phi = 1$ if the compound option is a call and -1 if a put.

$$\begin{aligned} & \phi\eta V e^{-\delta(T-t)} N_2(\phi\eta x, \eta y; \phi\rho) \\ & - \phi\eta M e^{-r(T-t)} N_2(\phi\eta x - \phi\eta\sigma_v\sqrt{\tau-t}, \eta y - \eta\sigma_v\sqrt{T-t}; \phi\rho) \\ & - \phi X e^{-r(\tau-t)} N_1(\phi\eta x - \phi\eta\sigma_v\sqrt{\tau-t}) \end{aligned}$$

where

$$x = \frac{\ln(Ve^{-\delta(\tau-t)}/V^*) + (r + \sigma_v^2/2)(\tau-t)}{\sigma_v\sqrt{\tau-t}}$$

$$y = \frac{\ln(Ve^{-\delta(T-t)}/M) + (r + \sigma_v^2/2)(T-t)}{\sigma_v\sqrt{T-t}}$$

$$\rho = \sqrt{(\tau-t)/(T-t)}.$$

The value V^* solves the equation

⁶The elasticity of S with respect to V is the percentage changed in S over the percentage change in V and can be expressed as $(\partial S/\partial V)(V/S)$.

$$\eta V^* e^{-\delta(T-\tau)} N_1(\eta z) - \eta M e^{-r(T-\tau)} N_1(\eta z - \sigma_v \sqrt{T-\tau}) - X = 0$$

$$z = \frac{\ln(V e^{-\delta(\tau-t)} / M) + (r + \sigma_v^2 / 2)(T - \tau)}{\sigma_v \sqrt{T - t}},$$

which, of course, is the Black-Scholes model.⁷

Geske (1977) has adapted his compound option pricing model to the case of coupon paying corporate bonds. Most importantly, however, Roll (1976) and Whaley (1979) have used the compound option model to obtain a closed-form solution for the price of an American call.

A variation of the compound option is the installment option. This is an option in which the premium is spread out in equal amounts over time. At each installment date, the holder of the option makes a decision about whether to exercise it, thereby paying the installment premium and continuing with the option. If the holder prefers not to continue, he simply fails to pay the installment. The option then terminates.⁸ This is very much like the compound option only there are typically several installments, necessitating a more complex option pricing model requiring the evaluation of higher order multivariate normal probability distributions. Also, the installments, which correspond to the exercise prices of the multiple underlying options are usually set in round numbers and typically they are all equal. Solving the pricing equation is quite difficult and usually requires a numerical solution. The installment option permits the holder to change his mind later and get out of the contract by simply failing to pay later installments. All previously paid installments are, of course, foregone.

References

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⁷ It is easy to see that by using the η factor, the basic Black-Scholes model can be written in a general form to accommodate both calls and puts.

⁸ Since this right is built into the contract, failure to pay an installment is not viewed as a default.

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