TEACHING NOTE 98-06:
RAINBOW (MIN-MAX) OPTION PRICING:

Version date: July 18, 2008 C:\Classes\Teaching Notes\TN98-06.doc

There are a number of variations of options in which there is more than one underlying asset. These options have a variety of interesting names. Nearly all of them are based on the maximum or minimum performer of two or more underlying assets or rates. The first paper to examine this type of option was Stulz (1982), who derived formulas for calls and puts that pay off based on which of two assets has the maximum or minimum value. We focus on the Stulz model and then show how it establishes a framework for variations of this type of option.

Suppose there are two assets whose current values are S_1 and S_2 and whose values at time T are S_{1T} and S_{2T}. Consider a call option with exercise price of X expiring at time T that pays off based on which of the two assets has the greater value. The payoff is written as

$$\text{Max}[\text{Max}(S_{1T},S_{2T}) - X,0]$$

and the current price is $c_{max}(S_1,S_2X,T)$, which we shall write more compactly as c_{max}. Likewise a call on the minimum has a current price of $c_{min}(S_1,S_2,X,T)$, which we write as c_{min}. The payoff of this call is

$$\text{Max}[\text{Min}(S_{1T},S_{2T}) - X,0]$$

Let us compare the payoffs of the call on the max with a portfolio consisting of a long call on asset 1, a long call on asset 2 and a short call on the min. We must consider two general cases, $S_{1T} < S_{2T}$ and $S_{1T} \geq S_{2T}$.

Case 1: $S_{1T} < S_{2T}$

	Payoff		
Current value of instrument	$S_{1T} < S_{2T} < X$	$S_{1T} \leq X \leq S_{2T}$	$X < S_{1T} < S_{2T}$
c_{max}	0	S_{2T} - X	S_{2T} - X
c_1	0	0	S_{1T} - X
c_2	0	S_{2T} - X	S_{2T} - X
-c_{min}	0	0	-(S_{1T} - X)
Total	0	S_{2T} - X	S_{2T} - X

Case 2: $S_{1T} \geq S_{2T}$

	Payoff		
Current value of instrument	$S_{2T} \leq S_{1T} < X$	$S_{2T} \geq X \leq S_{1T}$	$X < S_{2T} \leq S_{1T}$
c_{max}	0	S_{1T} - X	S_{1T} – X
c_1	0	0	S_{1T} – X
c_2	0	S_{1T} - X	S_{2T} - X
-c_{min}	0	0	-(S_{2T} - X)
Total	0	S_{1T} - X	S_{1T} – X

It is apparent that the call on the max is equivalent to the combination of long calls on both assets and a short call on the min. Therefore, the following relationship must hold for the current prices,

$$c_{max} = c_1 + c_2 - c_{min}.$$

This type of *max-min* parity will be useful for we need only derive a pricing model for one of the two min-max options and the price of the other can be obtained using the above equation.

Now let us consider a put on the minimum. Its current price is expressed as $p_{min}(S_1,S_2,X,T)$, which is written simply as p_{min}. Its payoff at expiration is

$$\text{Max}[X - \text{Min}(S_{1T},S_{2T}),0].$$

A put on the maximum has a current price of $p_{max}(S_1,S_2,X,T)$, which is written simply as p_{max}, and has a payoff of

$$\text{Max}[X - \text{Max}(S_{1T},S_{2T}),0].$$

Consider the following comparison whereby a put on the min is shown to be equivalent to a long position in risk-free bonds with face value X maturing at the option's expiration, a short position on a call on the min with zero exercise price and a long position on a call on the min with exercise price X.

Case 1: $S_{1T} < S_{2T}$

	Payoff		
Current value of instrument	$S_{1T} < S_{2T} < X$	$S_{1T} \le X \le S_{2T}$	$X < S_{1T} < S_{2T}$
p_{min}	$X - S_{1T}$	$X - S_{1T}$	0
Xe^{-rT}	X	X	X
$-c_{min\|strike = 0}$	$-S_{1T}$	$-S_{1T}$	$-S_{1T}$
c_{min}	0	0	$S_{1T} – X$
Total	$X - S_{1T}$	$X - S_{1T}$	0

Case 2: $S_{1T} \geq S_{2T}$

	Payoff		
Current value of instrument	$S_{2T} \leq S_{1T} < X$	$S_{2T} \leq X \leq S_{1T}$	$X < S_{2T} \leq S_{1T}$
p_{min}	$X - S_{2T}$	$X - S_{2T}$	0
Xe^{-rT}	X	X	X
$-c_{min\|strike=0}$	$-S_{2T}$	$-S_{2T}$	$-S_{2T}$
c_{min}	0	0	$S_{2T} - X$
Total	$X - S_{2T}$	$X - S_{2T}$	0

It is apparent that the portfolios are perfect substitutes. Consequently, their current values be the same, giving us the relationship,

$$p_{min} = Xe^{-rT} - c_{min|strike=0} + c_{min}.$$

Thus, having already obtained a formula for a call on the minimum, we can simply use that formula here, though note that in one case, we must use an exercise price of zero.

Finally we can easily obtain the price of the put on the maximum. Let us compare it to a long position in a pure discount bond, a short call on the max with an exercise price of zero and a long call on the max with an exercise price of X.

Case 1: $S_{1T} < S_{2T}$

	Payoff		
Current value of instrument	$S_{1T} < S_{2T} < X$	$S_{1T} \underset{\cdot}{\leq} X \leq S_{2T}$	$X < S_{1T} < S_{2T}$
p_{max}	$X - S_{2T}$	0	0
Xe^{-rT}	X	X	X
$-c_{max\|strike = 0}$	$-S_{2T}$	$-S_{2T}$	$-S_{2T}$
c_{max}	0	$S_{2T} - X$	$S_{2T} – X$
Total	$X - S_{2T}$	0	0

Case 2: $S_{1T} > S_{2T}$

	Payoff		
Current value of instrument	$S_{1T} \leq S_{2T} < X$	$S_{1T} \underset{\cdot}{\leq} X \leq S_{2T}$	$X < S_{1T} \leq S_{2T}$
p_{max}	$X - S_{1T}$	0	0
Xe^{-rT}	X	X	X
$-c_{max\|strike = 0}$	$-S_{1T}$	$-S_{1T}$	$-S_{1T}$
c_{max}	0	$S_{1T} - X$	$S_{1T} – X$
Total	$X - S_{1T}$	0	0

Consequently, the price of a put on the max can be obtained by the formula,

$$p_{max} = Xe^{-rT} - c_{max|strike=0} + c_{max}.$$

Once we price a call on the min, we can obtain the prices of the other options by these relationships. Incidentally, it might be tempting to argue that a call on the max with an exercise price of zero is equivalent to the asset itself. An ordinary call with an exercise price of zero always pays off the asset, but a call on the max with an exercise price of zero will always pay off the greater valued asset, which will not always be the same.

Pricing the Call on the Min

Now suppose our two assets follow the standard lognormal diffusions,

$$dS_1/S_1 = \mu_1 dt + \sigma_1 dz_1$$

$$dS_2/S_2 = \mu_2 dt + \sigma_2 dz_2,$$

where μ_i and σ_i are the drift and volatility of asset i, and dz_i is the Weiner process driving asset i, with i = 1,2. The correlation between assets 1 and 2 is ρ_{12}.

Let us form a hedge portfolio currently valued at V by placing x% of our wealth in asset 1, y% of our wealth in asset 2, and (100 - x - y)% of our wealth in the risk-free asset. For example, let x and y be expressed as percentages (e.g., x = .3, y = .6, 1 - x - y = .1). To express the current value of this portfolio consider that we invest xV dollars in asset 1. This will buy xV/S_1 shares. We invest yV dollars in asset 2, which will buy yV/S_2 shares. We then invest V - xV - yV dollars in the risk-free asset. The current value of our portfolio can, therefore, be expressed as

$$V = (xV/S_1)S_1 + (yV/S_2)S_2 + (1 - x - y)V.$$

The change in the value of this portfolio can be expressed as

$$dV = (xV/S_1)dS_1 + (yV/S_2)dS_2 + (1 - x - y)Vrdt,$$

which can be written as

$$dV = x(dS_1/S_1)V + y(dS_2/S_2)V + (1 - x - y)Vrdt.$$

Because V is driven by two stochastic processes, each of which follows the lognormal diffusion, we know that the change in V can also be expressed using the multivariate version of Itô's Lemma. In other words,

$$dV = \frac{\partial V}{\partial S_1}dS_1 + \frac{\partial V}{\partial S_2}dS_2 + \frac{\partial V}{\partial t}dt$$
$$+ \frac{1}{2}\left(\frac{\partial^2 V}{\partial S_1^2}S_1^2\sigma_1^2 + \frac{\partial^2 V}{\partial S_2^2}S_2^2\sigma_2^2 + 2\frac{\partial^2 V}{\partial S_1 \partial S_2}S_1 S_2 \sigma_1 \sigma_2 \rho_{12}\right)dt.$$

Suppose we select the value of x to be $(\partial V/\partial S_1)S_1/V$ and the value of y to be $(\partial V/\partial S_2)S_2/V$. Making these substitutions and setting these two equations equal to each other eliminates the stochastic terms, leaving the following non-stochastic partial differential equation.

$$\begin{aligned} Vr &= \frac{\partial V}{\partial S_1}S_1 r + \frac{\partial V}{\partial S_2}S_2 r + \frac{\partial V}{\partial t}dt \\ &+ \frac{1}{2}\left(\frac{\partial^2 V}{\partial S_1^2}S_1^2\sigma_1^2 + \frac{\partial^2 V}{\partial S_2^2}S_2^2\sigma_2^2 + 2\frac{\partial^2 V}{\partial S_1 \partial S_2}S_1 S_2 \sigma_1 \sigma_2 \rho_{12}\right)dt. \end{aligned}$$

We shall use this equation to replicate the payoff of a call on the minimum. Therefore its price, the solution to the above PDE, is subject to the following boundary conditions.

c_{min} = Max[min(S_{1T},S_{2T}) - X,0] at expiration

c_{min} = 0 if either asset is currently priced at zero

The second condition reflects the fact that if either asset is at zero, the asset value will never go above zero and that asset is therefore the minimum by default.[1] Since its price is always zero, the call can never be in-the-money. Stulz obtains the solution by discounting the expected payoff under risk neutrality. The formula is

$$\begin{aligned} c_{min} &= S_2 N_2\left(\gamma_1 + \sigma_2\sqrt{T}, \left(\ln(S_1/S_2) - (1/2)\sigma^2\sqrt{T}\right)/\sigma\sqrt{T}, \left(\rho_{12}(\sigma_1 - \sigma_2)\right)/\sigma\right) \\ &+ S_1 N_2\left(\gamma_2 + \sigma_1\sqrt{T}, \left(\ln(S_2/S_1) - (1/2)\sigma^2\sqrt{T}\right)/\sigma\sqrt{T}, \left(\rho_{12}(\sigma_2 - \sigma_1)\right)/\sigma\right) \\ &- Xe^{-rT}N_2\left(\gamma_{1,}\gamma_2, \rho_{12}\right) \end{aligned}$$

where

$$\begin{aligned} \gamma_1 &= \left(\ln(S_2/X) + (r - \sigma_2^2/2)T\right)/\sigma_2\sqrt{T} \\ \gamma_2 &= \left(\ln(S_1/X) + (r - \sigma_1^2/2)T\right)/\sigma_1\sqrt{T} \\ \sigma &= \sigma_1^2 + \sigma_2^2 - 2\rho_{12}\sigma_1\sigma_2 \end{aligned}$$

where σ is the volatility of a portfolio of a long position in asset 1 and a short position in asset 2.

There have been numerous extensions of the basic formula. Johnson (1987) develops the formula under the condition of more than two assets. Also several useful

---

[1] If an asset's price follows the lognormal diffusion, zero is an absorbing barrier. The price can never rise above zero, because the model defines all price changes relative to the current price. If the asset is a stock, a price of zero means that there is no possibility that the firm can generate any future value. If there were any such possibility, the price would have to be above zero or there would be an infinite number of arbitrage transactions coming from investors acquiring the asset for free with nothing to lose and something to potentially gain.

results are obtained by Rubinstein (1991). He first establishes a formula for an option that pays off the better of two risky assets or a fixed amount of cash. Letting X be the fixed amount of cash, we write this payoff as $\max(S_{1T},S_{2T},X)$. Rubinstein then derives the pricing formula for this option. Let us denote this price as c_{12X}. Then note the following relationship, $\max(S_{1T},S_{2T},X) - X = \max[0,\max(S_{1T},S_{2T}) - X]$. This implies that a long position in Rubinstein's option paying the best of two assets or X and a short position worth the present value of X is equivalent to a call on the max struck at X. Thus,

$$c_{12X} - Xe^{-rT} = c_{max}.$$

One particular problem encountered in using options on the max or min of two or more assets is that the asset values may be far apart at the start of the option. It would hardly be interesting to own a call paying off based on the maximum of two assets if one asset were currently worth \$100 and the other were currently worth \$20. To overcome this problem it is customary to express the option in terms of the assets' relative performances. For example, a call on the max would have a payoff as follows

$$\text{Max}[\text{Max}((S_{1T}-S_1)/S_1,(S_{2T}-S_2)/S_2) - X_r,0].$$

Here the rates of return of the two assets are compared and the payoff is determined by comparing the greater return to an exercise rate, X_r, expressed in terms of a return. To price this option, we first express the payoff as follows:

$$\text{Max}[\text{Max}((S_{1T}/S_1),(S_{2T}/S_2)) - (1+X_r),0].$$

This is the payoff of a call on the max in which the price of each asset has been normalized to a value of 1.0 at the start and the exercise price is expressed as 1 plus a return. This option can be valued directly with Stulz's formulas, inserting 1.0 as the price of each asset and using $1 + X_r$ as the exercise price. The volatilities and the correlation remain the same.[2]

Options on more than one asset are widely used in the over-the-counter options market. One particularly popular application is to have the assets be index returns on a sector of the market. Then the investor receives a return based on the better or worse performing sector. In practice options paying off based on more than one asset are

---

[2] The properties of the lognormal diffusion hold even if the asset is normalized to a value of 1.0, because the diffusion describes the proportional change in the asset's value.

sometimes called *two-color rainbow options*. One variation is the *outperformance* option, whose payoff would be as follows:

$$\text{Max}[0,(S_{1T}-S_1)/S_1 - (S_{2T}-S_2)/S_2].$$

Note that this option pays off the difference between the return on asset 1 and the return on asset 2, if that difference is positive, and zero if the difference is negative. Of course, the option could be structured with the assets reversed. There is no known closed-form solution for this type of option and numerical techniques are generally used. The standard version discussed above that pays off based on which asset has the better return and then subtracts the strike is commonly called an *alternative option*. Another variation is the *spread option*, whose payoff is

$$\max[0,(S_{1T} - S_{2T}) - X],$$

which has no known closed-form solution. Another variation is the *dual-strike option*, whose payoff is

$$\max[0,(S_{1T} - X_1),(S_{2T} - X_2)],$$

which also has no known closed-form solution.

References

Boyle, P. P. and X. S. Lin. "Bounds on Contingent Claims Based on Several Assets." *Journal of Financial Economics* 46 (1997), 383-400.

Boyle, P. P. and Y. K. Tse. "An Algorithm for Computing Values of Options on the Maximum or Minimum of Several Assets." *Journal of Financial and Quantitative Analysis* 25 (1990), 215-228.

Boyle, P., J. Evnine and S. Gibbs. "Numerical Valuation of Multivariate Contingent Claims." *Review of Financial Studies* 2 (1989), 241-250.

Broadie, M. and J. Detemple. "The Valuation of American Options on Multiple Assets." *Mathematical Finance* 7 (1997), 241-286.

Cheyette, O. "Pricing Options with Multiple Assets." *Advances in Futures and Options Research* 4 (1990), 69-81.

Gerber, H. U. and E. S. W. Shiu. "Martingale Approach to Pricing Perpetual American Options on Two Stocks." *Mathematical Finance* 6 (1996), 303-322.

Johnson, H. "Options on the Maximum or the Minimum of Several Risky Assets." *Journal of Financial and Quantitative Analysis* 22 (1987), 277-283.

Kamrad. B. and P. Ritchken. "Multinomial Approximating Models for Options with *k* State Variables." *Management Science* 12 (1991), 1640-1652.

Raymar, S. B. and M. J. Zwecher. "Monte Carlo Estimation of American Call Options on the Maximum of Several Stocks." *The Journal of Derivatives* 5 (Fall, 1997), 7-24.

Rich, D. R. and D. M. Chance. "An Alternative Approach to the Pricing of Options on Multiple Assets." *The Journal of Financial Engineering* 3 (1993), 271-286.

Rubinstein, M. "Return to Oz." 7 *Risk* (November, 1994), 67-71.

Rubinstein, M. "Somewhere Over the Rainbow." *Risk* 4 (November, 1991), 61-63.

Stulz, R. M. "Options on the Minimum or the Maximum of Two Risky Assets." *Journal of Financial Economics* 10 (1982), 161-181.