

TEACHING NOTE 99-02:

DERIVATION AND INTERPRETATION OF THE BLACK-SCHOLES MODEL

Version date: July 18, 2008

C:\Classes\Teaching Notes\TN99-02.doc

From the basic principles associated with the standard stochastic process used in modeling asset prices and an understanding of Itô's Lemma, we can now derive the Black-Scholes model for pricing European options. Let the asset price follow the standard lognormal diffusion process given by the stochastic differential equation known as Geometric Brownian Motion,

$$dS_t = S_t \alpha dt + S_t \sigma dz_t, \quad (1)$$

where S_t is the asset price at time t , dS_t is the change in the asset price per unit of time dt , α is the drift or expected rate of return on the asset, σ is the volatility of the return on the asset and dz_t is the Wiener process commonly used to generate the uncertainty. Recall that $dz_t = \varepsilon_t \sqrt{dt}$ where ε_t is a standard normal random variable (mean 0, variance 1) and that the properties of dz_t are that $E(dz_t) = 0$, $\text{var}(dz_t) = dt$ and that $dz_t^2 = dt$.

Consider a European call option with exercise price X . The option price is assumed to be a function of only two variables, the asset price and time.¹ Thus, we write the option price function in its general form as $c(S_t, t)$ and more loosely as c_t . The option's time to expiration is $\tau = T - t$. At expiration, the option price is $c_T = \text{Max}(0, S_T - X)$.

Let us construct a portfolio consisting of h units of the asset and one short call option. The value of the portfolio at time t is

$$V_t = hS_t - c_t. \quad (2)$$

The value h could be subscripted with a t ; its value is set and known at t . We hold h_t units of the asset per one short call. As we move forward, however, the value of our holdings changes due to changes in the value of the asset and the call. We are still holding h units of the asset and one short call. As noted later, we shall have to reset h_t according to the new market conditions. Thus, h_t is set at t and remains constant until we rebalance, which occurs after we determine how

¹Naturally the option price is also a function of the exercise price, X , the risk-free rate, r , and the volatility, σ , but these values are not allowed to vary and, thus, they are parameters rather than variables. They can, of course, vary from problem to problem, but they are not allowed to vary internally, i.e., in the dynamics of the evolution of the underlying asset and the price of the specific option over the life of the option.

our portfolio has performed. Consequently, we can treat h_t as a constant and we denote it as h .

Our objective is to make this portfolio risk-free. Since the call price is a function of S_t , which follows the stochastic differential equation in (1), and t , we use Itô's Lemma to express the call price change in terms of the changes in S_t and t :²

$$dc = \frac{\partial c}{\partial S} dS + \frac{\partial c}{\partial t} dt + \frac{1}{2} \frac{\partial^2 c}{\partial S^2} dS^2.$$

Given Equation (1) and the properties of the Wiener process, dz , we know that $dS^2 = S^2 \sigma^2 dt$. From here on we shall make this substitution whenever we use the above result from Itô's Lemma.

We know that the value of the portfolio is a function of the asset price and the option price. Hence, using the total differential rule, we can express the change in the value of the portfolio as

$$dV = \frac{\partial V}{\partial S} dS + \frac{\partial V}{\partial c} dc.$$

Given Equation (2), we can determine the partial derivatives as $\partial V / \partial S = h$ and $\partial V / \partial c = -1$.

Thus, the change in the value of the portfolio is

$$dV = h dS - dc. \quad (3)$$

Substituting the result from Itô's Lemma for dc into (3) gives

$$dV_t - h dS - \left(\frac{\partial c}{\partial S} dS + \frac{\partial c}{\partial t} dt + \frac{1}{2} \frac{\partial^2 c}{\partial S^2} S^2 \sigma^2 dt \right). \quad (4)$$

Equation (4) is a differential equation that expresses the change in the value of the portfolio in terms of a number of expressions. Note that the stochastic terms are the dS terms. We are free, however, to set h to whatever we want. By setting h to $\partial c / \partial S$, the two dS terms cancel. This leaves the following expression for the change in the value of the portfolio:

$$dV = - \left(\frac{\partial c}{\partial t} dt + \frac{1}{2} \frac{\partial^2 c}{\partial S^2} S^2 \sigma^2 dt \right). \quad (5)$$

Since there are no stochastic terms, this portfolio is perfectly hedged. Thus, its value, V , must increase at the risk-free rate. This specification is made by the requirement that

$$dV = Vr dt.$$

We now substitute $hS - c = (\partial c / \partial S) S - c$ in the above equation for V to obtain

$$dV = \left(\frac{\partial c}{\partial S} S - c \right) r dt . \quad (7)$$

We now have two expressions for dV , Equations (5) and (7). Equating these and canceling all dt terms gives,

$$rS \frac{\partial c}{\partial S} + \frac{\partial c}{\partial t} + \frac{1}{2} \frac{\partial^2 c}{\partial S^2} \sigma^2 S^2 = rc . \quad (8)$$

This is a partial differential equation of which the solution is the call option price. This equation, or a variation of it, appears often in derivative pricing theory. The solution is determined by the boundary conditions, which refers to the value of c at the option expiration. In the case of a European call, the condition is that $c_T = \text{Max}(0, S_T - X)$.

Equation (8), a second-order parabolic partial differential equation, is actually a variation of a famous equation in physics that models the transfer of heat. The solution procedure is well-known to physicists and one version of it is presented in Appendix A. Black and Scholes (1973) of M.I.T. first obtained the solution by taking advantage of previous research on option pricing that gave an idea of what the solution would look like. Bachelier (1900) had derived the solution under Arithmetic Brownian Motion.³ Sprenkle (1964) was the first to assume a lognormal distribution, i.e., Geometric Brownian Motion, but Sprenkle attempted to adjust for risk, which was neither necessary nor correct. In addition, he assumed a zero interest rate. Boness (1964) added a non-zero interest rate but also improperly adjusted for risk. Samuelson (1965) also incorrectly attempted to adjust for risk. Nonetheless, each of these formulas resembles the correct solution as found by Black and Scholes.

Black and Scholes, however, originally derived the equation by using the Capital Asset Pricing Model, which provides the equation for the expected return on a risky asset as a function of its risk. Though Black had a Ph.D. in applied mathematics from Harvard, he was not aware that the differential equation was the heat transfer equation. It is rumored that someone else noticed and showed him how to obtain the solution. At approximately the same time, however, a colleague of Black's, Robert Merton (1973), also of M.I.T., wrote a paper developing the model with virtually the full mathematical solution. It is not clear whether Merton or Black and Scholes came up with the insight that the option price could be derived by forming a

²To reduce notational clutter, where feasible we eliminate the t subscripts on c , V and S .

³In Arithmetic (accent on third syllable) Brownian Motion, the asset follows the stochastic process, $dS = \alpha dt + \sigma dz$. In such a case, the asset value can go below zero. In the Geometric Brownian Motion that we use, the rate of change of the asset follows

continuously adjusted risk-free hedge but Black and Scholes have without doubt received the most credit.⁴ The model should probably be called the Black-Scholes-Merton model but Merton has never received full credit. All three scholars, however, became famous and wealthy. Merton remained in academia but commands hefty consulting fees and is involved in a major investment management firm. Scholes left academia for awhile but has remained affiliated with Stanford University and is also involved in professional investment management. Black left academia in 1983 for Goldman Sachs and was made a partner. He continued to produce scholarly research up until his unfortunate death from throat cancer in 1995 at the age of 57.

The solution to the equation is known as the Black-Scholes option pricing model,⁵

$$c = SN(d_1) - Xe^{-r\tau}N(d_2), \quad (9)$$

where

$$d_1 = \frac{\ln(S/X) + (r + \sigma^2/2)\tau}{\sigma\sqrt{\tau}},$$

and

$$d_2 = d_1 - \sigma\sqrt{\tau}.$$

The value $N(d_i)$, $i=1,2$, is the cumulative normal probability,

$$N(d_1) = \int \frac{1}{\sqrt{2\pi}} e^{-x^2/2} dx.$$

One method for deriving the solution by solving the partial differential equation is provided in Appendix A.⁶

The model for pricing a put is easily derived from put-call parity. We know that $p = c - S$

the stochastic process. By having S on the right-hand side, if the value of S ever becomes zero, then dS is permanently zero.

⁴Full details of this interesting side story are not clear but it may be the case that Merton developed the insight of the risk-free hedge. Merton, however, in deference to Black and Scholes, withheld the publication of his paper until the Black and Scholes paper appeared in print. Interestingly, Black and Scholes were working on two papers at the time, with the second paper being an empirical test of the model using over-the-counter option price data. That paper was presented at the annual meeting of the American Finance Association in 1971 and appeared in print in *The Journal of Finance* in 1972. Meanwhile, the original Black and Scholes paper, which derived the model, was rejected by the University of Chicago's renowned *Journal of Political Economy*. Black and Scholes then submitted the paper to Harvard's *Review of Economics and Statistics*, which also rejected it. Then the University of Chicago's Merton Miller suggested that the *Journal of Political Economy* reconsider the paper, because Miller predicted it was a major breakthrough. The *JPE* then agreed to publish the paper and it appeared in print in 1973, around the same time as the Merton paper but after the model's empirical tests had already appeared in the *Journal of Finance*.

⁵Rubinstein (1976) has shown that the Black-Scholes model can be derived an alternative way, which requires no assumptions about the ability to continuously adjust a hedge.

⁶For the case where the asset is assumed to pay a constant continuously compounded yield of δ , the solution differs only in that the asset price S is replaced by the asset price less the present value of the payments over the life of the option, $Se^{-\delta(T-t)}$. The derivation proceeds under the assumption that the asset price drift is $(\alpha - \delta)dt$. If the payments occur at discrete times it is necessary to find the present value of the payments, obtained by discounting at the risk-free rate, and subtract this from the drift. The final solution is still of the same general form as Equation (10) but S is replaced by S minus the present value of the discrete

+ $Xe^{-r\tau}$, and we can substitute the Black-Scholes formula for c to obtain⁷

$$p = Xe^{-r\tau}N(-d_2) - SN(-d_1). \quad (10)$$

Once the Black-Scholes equation is found, it is important to determine the partial derivatives, some of which should then be substituted back into the partial differential equation to verify that the solution is correct. Thus, in the next section we examine the comparative statics of the Black-Scholes call option pricing model.

Comparative Statics

The derivatives of the Black-Scholes model are not easy to obtain. The following results will be useful:

$$\frac{\partial N(d_1)}{\partial d_1} = \frac{1}{\sqrt{2\pi}} e^{-d_1^2/2}, \quad (11)$$

$$\frac{\partial N(d_2)}{\partial d_2} = \frac{1}{\sqrt{2\pi}} e^{-(d_1 - \sigma\sqrt{\tau})^2/2}, \quad (12)$$

$$S = Xe^{d_1\sigma\sqrt{\tau} - (r + \sigma^2/2)\tau}. \quad (13)$$

The partial derivative with respect to the asset price, usually called the option's *delta*, is found as follows

$$\frac{\partial c}{\partial S} = S \frac{\partial N(d_1)}{\partial d_1} \frac{\partial d_1}{\partial S} + N(d_1) - Xe^{-r\tau} \frac{\partial N(d_2)}{\partial d_2} \frac{\partial d_2}{\partial S}.$$

Using (11), (12) and (13), gives

$$\frac{\partial c}{\partial S} = N(d_1) \geq 0. \quad (14)$$

Clearly the delta is like a probability as it ranges from zero to one. We shall see later, however, that it does not have a simple probability interpretation.

The second derivative with respect to the asset price, called the option's *gamma*, is found as follows:

payments.

⁷ Alternatively the put option pricing model can be derived by setting up a risk-free hedge involving the holding of one unit of the asset and h puts. The derivation would proceed exactly as in the call option pricing model derivation, except that the boundary condition would be $p_T = \text{Max}(0, X - S_T)$. Also, the hedge ratio h would be $1/(\partial p/\partial S)$ and $\partial p/\partial S$ would (from put-call parity) equal $\partial c/\partial S - 1$.

$$\begin{aligned}\frac{\partial^2 c}{\partial S^2} &= \frac{\partial}{\partial S} \left(\frac{\partial c}{\partial S} \right) = \frac{\partial N(d_1)}{\partial d_1} \frac{\partial d_1}{\partial S}, \\ &= \frac{1}{S\sigma\sqrt{\tau}} \frac{\partial N(d_1)}{\partial d_1} \geq 0,\end{aligned}\tag{15}$$

where the value $\partial N(d_1)/\partial d_1$ is given in (11). The gamma shows how the delta changes as the asset price changes. This result is useful since a risk-free return is guaranteed only if the hedge is maintained at the appropriate level of delta. If the gamma is large, then the delta will change by a large amount when the asset price changes and hedging will be more difficult. The gamma will tend to be large when the option is closest to being at-the-money.

Another useful result, though not needed to verify the solution, is the partial of the delta with respect to time.

$$\begin{aligned}\frac{\partial^2 c}{\partial \tau \partial S} &= \frac{\partial}{\partial \tau} \left(\frac{\partial c}{\partial S} \right) = \frac{\partial N(d_1)}{\partial d_1} \frac{\partial d_1}{\partial \tau}, \\ &= \frac{\partial N(d_1)}{\partial d_1} \frac{\sigma\sqrt{\tau}(r + \sigma^2/2) - (\ln(S/X) - (r + \sigma^2/2)\tau)\sigma/(2\sqrt{\tau})}{\sigma^2\tau} \geq 0,\end{aligned}\tag{16}$$

The sign of (16) can be determined as follows:

$$\frac{\partial^2 c}{\partial \tau \partial S} \geq 0 \text{ as } S/X \geq e^{(r+\sigma^2/2)\tau}.$$

This results shows that the delta will increase, moving toward one, as expiration nears if the call is in-the-money and decrease, moving toward zero, as expiration nears if the call is out-of-the-money. Further analysis would reveal that the gamma would approach infinity as expiration nears if the option is nearly at-the-money.

The derivative with respect to time to expiration is found as

$$\frac{\partial c}{\partial \tau} = S \frac{\partial N(d_1)}{\partial d_1} \frac{\partial d_1}{\partial \tau} - Xe^{-r\tau} \frac{\partial N(d_2)}{\partial d_2} \frac{\partial d_2}{\partial \tau} + rN(d_2)Xe^{-r\tau}.$$

Using (11), (12), and (13) and the fact that $\partial d_2/\partial \tau = \partial d_1/\partial \tau - \sigma/(2\sqrt{\tau})$, we have

$$\frac{\partial c}{\partial \tau} = \frac{S\sigma}{2\sqrt{\tau}} \frac{\partial N(d_1)}{\partial d_1} + rXe^{-r\tau}N(d_2) \geq 0.\tag{17}$$

This means that the longer the time to expiration, the more valuable the option. The value $\partial c/\partial \tau$, which is $-\partial c/\partial t$ is often preferred in practice and is known as the option's *theta*.

Once we have the three derivatives $\partial c/\partial S$, $\partial c/\partial t$, and $\partial^2 c/\partial S^2$, we then insert these values

into the partial differential equation and check the solution. We shall do that a few paragraphs down. For now, however, let us finish taking the derivatives.

The partial derivative with respect to the exercise price is

$$\frac{\partial c}{\partial X} = S \frac{\partial N(d_1)}{\partial d_1} \frac{\partial d_1}{\partial X} - X e^{-rT} \frac{\partial N(d_2)}{\partial d_2} \frac{\partial d_2}{\partial X} + e^{-rT} N(d_2).$$

Taking $\partial d_1/\partial X$, $\partial d_2/\partial X$ and using (11), (12), and (13), gives

$$\frac{\partial c}{\partial X} = e^{-rT} N(d_2) \leq 0. \quad (18)$$

Of course the exercise price does not really change so this expression means only that a call option with an infinitesimally lower exercise price than another call option would have a higher price.⁸

With knowledge of the values, $\partial c/\partial S$ and $\partial c/\partial X$, we can obtain another interesting result. It can be observed that the payoff of a standard call option is linearly homogeneous with respect to the asset price and exercise price. This result is easily seen by noting the payoff, $c_T = \text{Max}(0, S_T - X)$. Multiplying the payoff by any constant such as k , we obtain $\text{Max}(0, kS_T - kX)$, which is easily seen to be simply kc_T . If the payoff of an option is linearly homogeneous, then its price at a time prior to expiration must also be linearly homogeneous. This result is by virtue of the fact that any asset's price is by definition the discounted value of its expected future value. Though we have not yet addressed the question of the appropriate discount rate or how to take expectations in this type of problem, we nonetheless know that discounting and taking expectations are both linear operations. When applied to a linearly homogeneous function, linear operations do not alter the property of linear homogeneity. Thus, by application of Euler's Theorem, a linearly homogeneous function such as c can be expressed as

$$c = \frac{\partial c}{\partial S} S + \frac{\partial c}{\partial X} X.$$

Substituting the values of the partials from equations (14) and (18), we obtain the Black-Scholes model, Equation (9). Noting this result, we see that replication of a long position in a call, as indicated by a positive unit value as the coefficient on c on the left-hand side, is obtained at the current instant t by holding $\partial c/\partial S = N(d_1)$ units of the asset and bonds in the amount of $\partial c/\partial X = -e^{-rT} N(d_2)$. The minus sign indicates that the bonds are actually short, representing borrowing.

⁸There are, however, options in which the exercise price can change and these are surprisingly easy to price in the framework used here.

In other words have borrowings of $N(d_2)$ bonds with current value $Xe^{-r\tau}$ and maturity value X .

Returning to our comparative statics, the partial derivative with respect to the standard deviation is

$$\frac{\partial c}{\partial \sigma} = S \frac{\partial N(d_1)}{\partial d_1} \frac{\partial d_1}{\partial \sigma} - Xe^{-r\tau} \frac{\partial N(d_2)}{\partial d_2} \frac{\partial d_2}{\partial \sigma}.$$

Using (11), (12), and (13) and the fact $\partial d_2 / \partial \sigma = \partial d_1 / \partial \sigma - \sqrt{\tau}$ gives

$$\frac{\partial c}{\partial \sigma} = S\sqrt{\tau} \frac{\partial N(d_1)}{\partial d_1} \geq 0. \quad (19)$$

This value is called the option's *vega*.⁹ In practice the volatility of a given asset does change from time to time during the life of an option and this formula is often used to gauge the sensitivity of the option price to a volatility change. One must be careful, however, to recognize that the Black-Scholes framework does not really permit the volatility to change. To be technically correct, we must interpret the vega as the differences in the prices of two otherwise identical options that have infinitesimally different volatilities.¹⁰ The vega tends to be fairly large for most options, reflecting the fact that an option's price is typically quite sensitive to the volatility input.

The partial derivative with respect to the risk-free rate, called the *rho*, is found as

$$\frac{\partial c}{\partial r} = S \frac{\partial N(d_1)}{\partial d_1} \frac{\partial d_1}{\partial r} - Xe^{-r\tau} \frac{\partial N(d_2)}{\partial d_2} \frac{\partial d_2}{\partial r} + \tau N(d_2)Xe^{-r\tau}.$$

Taking $\partial d_1 / \partial r$ and using (11), (12), and (13) gives

$$\frac{\partial c}{\partial r} = X\tau N(d_2)e^{-r\tau} \geq 0. \quad (20)$$

Like the volatility, the interest rate is technically not permitted to change during the life of the option. Thus, rho is interpreted only as the difference between an option price if an infinitesimally different risk-free rate is used. Typically the rho is quite small for most options, reflecting the fact that interest rates do not impart a particularly large effect on the option price.¹¹

The partial derivatives for the Black-Scholes put option pricing model can be obtained by using the put-call parity equation and the partial derivatives for the call. They are provided in

⁹ It has also been known as the option's *kappa* or *lambda*. The partial derivatives are often called the option "Greeks," though vega is not a Greek word.

¹⁰ There are option pricing models that permit changing volatility but these are advanced topics that we do not cover in these notes.

¹¹ Obviously this would change for options on bonds, interest rates or related instruments but we do not cover these here.

Appendix B.

Verifying the Solution to the Partial Differential Equation

If the Black-Scholes formula (Equation (9)) is the correct solution to the partial differential equation (Equation (8)), then we should be able to insert the derivatives $\partial c/\partial S$, $\partial^2 c/\partial S^2$, and $\partial c/\partial t$ into Equation (8) and obtain Equation (9). Making these substitutions reveals that the solution is indeed correct. The boundary condition must also be checked. Letting $\tau = 0$, then the Black-Scholes formula must converge to $c_T = \text{Max}(0, S_T - X)$:

If $S_T > X$

$d_1 \rightarrow \infty$, meaning that $N(\infty) \rightarrow 1$

$d_2 \rightarrow \infty$, meaning that $N(\infty) \rightarrow 1$

so $c_T \rightarrow S_T - X$.

If $S_T \leq X$

$d_1 \rightarrow \infty$, meaning that $N(\infty) \rightarrow 0$

$d_2 \rightarrow \infty$, meaning that $N(\infty) \rightarrow 0$

so $c_T \rightarrow 0$.

So we see that the Black-Scholes model converges in the limit to the boundary conditions as specified by the value of the option at expiration.

An Alternative Derivation

Another approach to deriving the Black-Scholes model is to treat the option as if it were valued in a world of risk neutral investors. The Black-Scholes price is then found as the present value of the expected call price at expiration. In such a world, investors are not concerned about risk and the risk-free rate is an appropriate discount rate. Thus,

$$c = E[c_T]e^{-r\tau}.$$

The assumption of risk neutrality may appear to be unpalatable but it is entirely appropriate in option pricing. We should have already noticed that the expected return on the asset is not a factor in the Black-Scholes model. The risk preferences of investors are reflected in an asset's expected return; thus, the Black-Scholes model does not require a term for investor risk preferences. That does not mean that investors are actually risk neutral. On the contrary, nearly all investors are risk averse, but their feelings about risk are reflected in the asset price. Risk is priced in the market for the underlying asset and the price of risk, what we call the *risk*

premium, shows up in the price of the asset. All other things equal, the price of the asset is lower if there is more risk or if the degree of risk aversion of investors increases.

More precisely, however, we define this operation as taking discounted expectations under an *equivalent martingale probability distribution*. The price of any asset that trades in a no-arbitrage world can be obtained as the discounted expectation of its future payoff, where the probabilities that determine its expectation are called equivalent martingale probabilities and the discount rate is the risk-free rate. A martingale is a stochastic process in which the expectation of an asset's future value is its current value. As has been shown elsewhere, a world of no-arbitrage is equivalent to a world in which the asset price can be obtained using the probabilities that would exist if investors were risk neutral; discounting then proceeds at the risk-free rate. In option pricing we are confident that no arbitrage opportunities exist and we can take investors' feelings about risk as given and reflected already in the asset price. No further adjustment for risk is required and indeed any further adjustment would be a double penalty. Let us state more formally that our option price is

$$c = \{E_t^Q[c_T]\}e^{-rt},$$

which means that the expectation of the option payoff, c_T , is taken at time t using equivalent martingale probabilities and discounted to the present at the risk-free rate.

Recall from our model that $dz = \varepsilon\sqrt{dt}$. The time increment over the life of the option is $\Delta t = \tau$ and $\Delta z = \varepsilon^* \sqrt{\Delta t} = \varepsilon^* \sqrt{\tau}$. Recall the original stochastic process, $dS = S\alpha dt + S\sigma dz$. Given that ε is normally distributed and is the source of all of the uncertainty, we should be able to use normal probability theory to evaluate the above expectation. We shall need to express the stochastic process in such a manner that the return on the asset is normally distributed. In Geometric Brownian Motion the log return on the asset is normally distributed, so we will need the stochastic process for the log of the asset return.

Define $dS + S$ as the asset price at an instant, dt , later. Thus, we can write the stochastic process as $dS + S = S[1 + \alpha dt + \sigma dz]$. Working with the term in brackets, note that we can write it in the following, seemingly complex, way:

$$1 + \alpha dt + \sigma dz = 1 + \left[(\alpha - \sigma^2/2)dt + \sigma dz + \left((\alpha - \sigma^2/2)dt + \sigma dz \right)^2 / 2 \right].$$

By multiplying out the terms on the right hand side, it is easy to verify that the above statement is true. Now define $\mu = \alpha - \sigma^2/2$ and the above can be written as $1 + [\mu dt + \sigma dz + (\mu dt + \sigma dz)^2/2]$.

The term in brackets is equivalent to a second-order Taylor series expansion of the function $e^{\mu dt + \sigma dz}$. A second-order expansion is sufficient, because all terms higher than second order will involve powers of dt greater than 1.0.

Now we can write out the stochastic process as $dS + S = S e^{\mu dt + \sigma dz}$. Dividing by S we obtain $dS/S + 1 = e^{\mu dt + \sigma dz}$. Taking natural logs, we have the stochastic process of the log return on the asset,

$$\ln\left(\frac{dS + S}{S}\right) = \mu dt + \sigma dz.$$

This result confirms that the log return is normally distributed with mean μ and volatility σ . For our purposes here, we use the following version,

$$dS + S = S^{\mu dt + \sigma dz},$$

Now we can proceed with our derivation. Noting that the time increment until expiration is τ , we have the asset price at expiration as $S_{t+\tau} = S_T$ and the stochastic process for z as $\Delta z(T) = \varepsilon^* \sqrt{\tau}$. Thus,

$$S_T = S_t e^{\mu\tau + \sigma\varepsilon^* \sqrt{\tau}}$$

with $\Delta z(T)$ normally distributed with mean zero and variance τ , per the central limit theorem.

Our objective is to evaluate the expectation of c_T . By definition,

$$E_t^Q(c_T) = \int_X^\infty (S_T - X) f(S_T) dS_T.$$

Note that we integrate only over the interval (X, ∞) because $c_T = 0$ for $0 \leq S_T \leq X$. Thus,

$$\begin{aligned} E_t^Q(c_T) &= \int_X^\infty S_T f(S_T) dS_T - \int_X^\infty X f(S_T) dS_T \\ &= \int_X^\infty S_T f(S_T) dS_T - X \text{Pr ob}(S_T > X). \end{aligned}$$

What we want is the expected value of S_T given that $S_T > X$ minus the expected payout of the exercise price, i.e., the exercise price times the probability that the option will be exercised. That is,

$$E_t^Q(c_T) = E_t^Q[S_T | S_T > X] - X \text{Pr ob}^Q[S_T > X].$$

Let us first evaluate the second term on the right-hand side. By definition,

$$\Pr \text{ob}^Q[S_T > X] = \Pr \text{ob}^Q[Se^{\mu\tau + \sigma\epsilon^* \sqrt{\tau}} > X].$$

Note that $Se^{\mu\tau + \sigma\epsilon^* \sqrt{\tau}} > X$ is equivalent to $\ln(S/X) + \mu\tau + \sigma\epsilon^* \sqrt{\tau}$ or

$$\epsilon^* > -\frac{[\ln(S/X) + \mu\tau]}{\sigma\sqrt{\tau}}.$$

Recall that α is the expected simple return on the asset and μ is the expected logarithmic return on the asset where $\mu = \alpha - \sigma^2/2$. Under the equivalent martingale/risk neutrality approach, we can let $\alpha = r$ so that $\mu = r - \sigma^2/2$ so that

$$\epsilon^* > -\frac{[\ln(S/X) + (r - \sigma^2/2)\tau]}{\sigma\sqrt{\tau}}.$$

You should recognize this as $\epsilon^* > -d_2$ or $\epsilon^* < d_2$. So our second term in (21) is

$$X(\Pr \text{ob}^Q[S_T > X]) = XN(d_2).$$

Now we look at the first term on the right-hand side of (21). It can be written as

$$\begin{aligned} \int_X^\infty S_T f(S_T) dS_T &= \int_X^\infty Se^{\mu\tau + \sigma\epsilon^* \sqrt{\tau}} f(S_T) dS_T \\ &= Se^{\mu\tau} \int_X^\infty e^{\sigma\epsilon^* \sqrt{\tau}} f(S_T) dS_T. \end{aligned}$$

We previously showed that $S_T > X$ implies that $\epsilon^* > -d_2$ so we can change the variable of integration giving us the equivalent statement,

$$Se^{\mu\tau} \int_{-d_2}^\infty e^{\sigma\epsilon^* \sqrt{\tau}} f(\epsilon^*) d\epsilon^*.$$

Since ϵ^* is normally distributed we can substitute the density for the normal distribution,

$$\begin{aligned} Se^{\mu\tau} \int_{-d_2}^\infty e^{\sigma\epsilon^* \sqrt{\tau}} \left(e^{-\epsilon^{*2}/2} / \sqrt{2\pi} \right) d\epsilon^* \\ = Se^{\mu\tau} \int_{-d_2}^\infty \left(e^{-\epsilon^{*2}/2 + \sigma\epsilon^* \sqrt{\tau}} / \sqrt{2\pi} \right) d\epsilon^*. \end{aligned}$$

By completing the square in the exponent, we obtain

$$Se^{\mu\tau} \int_{-d_2}^\infty e^{-\left(\epsilon^* - \sigma\sqrt{\tau}\right)^2/2 + \sigma^2\tau/2} / \sqrt{2\pi} d\epsilon^*.$$

Now we make a change of variables. Let $y = \epsilon^* - \sigma\sqrt{\tau}$ so that $\epsilon^* = \sigma\sqrt{\tau} + y$. This changes the

lower limit of integration. Formerly we were interested in the value of ε^* over the range $(-d_2, \infty)$. Since $d_2 = d_1 - \sigma\sqrt{\tau}$, this means that $\varepsilon^* > -(d_1 - \sigma\sqrt{\tau})$ or $\varepsilon^* > \sigma\sqrt{\tau} - d_1$. Substituting $y + \sigma\sqrt{\tau}$ for ε^* means that $y + \sigma\sqrt{\tau} > \sigma\sqrt{\tau} - d_1$; thus, $y < d_1$. Now our expression can be written as

$$Se^{\mu\tau + \sigma^2\tau/2} \int_{-\infty}^{d_1} e^{-y^2/2} dy.$$

The integral is simply the value $N(d_1)$. Since $\mu + \sigma^2/2 = r$, our first term can be written as $Se^{r\tau}N(d_1)$. Thus,

$$E_T^Q(c_T) = Se^{r\tau}N(d_1) - XN(d_2).$$

The value of the call today is the present value of $E_T^Q[c_T]$ obtained using the factor $e^{-r\tau}$. Thus,

$$c = SN(d_1) - Xe^{-r\tau}N(d_2).$$

which is the Black-Scholes formula.

This derivation makes it a little easier to see the proper interpretation of the normal probabilities, $N(d_1)$ and $N(d_2)$. $N(d_2)$ is the probability that the call will be exercised provided one assumes that the asset drift is the risk-free rate. Of course, any risky asset will command a risk premium so the actual drift is α , which is higher than r by the risk premium. Consequently, the actual probability of exercise is higher than $N(d_2)$. $N(d_1)$, however, does not lend itself to a simple probability interpretation. $SN(d_1)$ is correctly interpreted as the present value, using the risk-free interest rate, of the expected asset price at expiration, given that the asset price at expiration is above the exercise price. Most people evade the answer by stating that $N(d_1)$ is just the option's delta.

Appendix A. Mathematical Details of the Black-Scholes Solution

The mathematical details that lead from the partial differential equation (Equation (8)) to the Black-Scholes formula (Equation (9)) are seldom seen in print. One exception is Kutner (1988). This appendix is adapted from that article. There are a few other sources of derivations, which are listed in the references, that take a slightly different approach.

Solving a differential equation frequently involves guessing at the form of the solution. Some clues as to the correct form were provided in the research on options that preceded Black and Scholes. Let the solution be written generally as $c_t = e^{-r\tau}y$ where $y = f(u, q)$. The variables u and q are specified as

$$u = \left(2/\sigma^2\right)\left(r - \sigma^2/2\right)\left[\ln\left(S/X\right) - \left(r - \sigma^2/2\right)(-\tau)\right]$$

$$q = -\left(2/\sigma^2\right)\left(r - \sigma^2/2\right)^2(-\tau).$$

The partials of $c = e^{-r\tau}y(u,q)$ are found as

$$\frac{\partial c}{\partial S} = e^{-r\tau}y_u\left(2/\sigma^2\right)\left(r - \sigma^2/2\right)/S$$

$$\frac{\partial c}{\partial t} = e^{-r\tau}\left[-y_u\left(2/\sigma^2\right)\left(r - \sigma^2/2\right)^2 - y_q\left(2/\sigma^2\right)\left(r - \sigma^2/2\right)^2 + ry\right]$$

$$\frac{\partial^2 c}{\partial S^2} = e^{-r\tau}\left[y_u\left(2/\sigma^2\right)\left(r - \sigma^2/2\right)y_u\left(1/S^2\right) - y_q\left(2/\sigma^2\right)^2\left(r - \sigma^2/2\right)^2\left(1/S^2\right)y_{uu}\right],$$

where y_u , y_q and y_{uu} are shorthand notation for $\partial y/\partial u$, $\partial y/\partial t$ and $\partial^2 y/\partial u^2$. Note that $\partial t = -\partial\tau$.

Substituting into the Black-Scholes PDE gives

$$rSe^{-r\tau}y_u\left(2/\sigma^2\right)\left(r - \sigma^2/2\right)/S_t - rc_t$$

$$+ e^{-r\tau}\left[y_u\left(2/\sigma^2\right)\left(r - \sigma^2/2\right)^2 - y_q\left(2/\sigma^2\right)^2\left(r - \sigma^2/2\right)^2 + ry\right]$$

$$+ \frac{1}{2}e^{-r\tau}\left[y_u\left(2/\sigma^2\right)\left(r - \sigma^2/2\right)y_u\left(1/S^2\right) + \left(2/\sigma^2\right)^2\left(r - \sigma^2/2\right)^2\left(1/S^2\right)y_{uu}\right] = 0,$$

which simplifies to

$$y_q = -y_{uu}.$$

This is the heat exchange equation of thermodynamics. The solution is well-known (Black and Scholes site Churchill (1963, pp. 154-155). If V is a function of x and t and $V_t = KV_{xx}$ for $-\infty < x < \infty$, $t > 0$ and $V = f(x)$ when $t = 0$ for $-\infty < x < \infty$, the solution is of the form

$$V = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} f(x + 2\eta\sqrt{kt}) e^{-\eta^2} d\eta.$$

In our problem V is the option price, x is u , t is q , $k = 1$, and $f(x)$ is the boundary condition. The variable q can equal zero only at expiration ($\tau = 0$). We need to find the function $f(x)$ consistent with the option's expiration value:

$$y(u,0) = c(S_T,0) = S_T - X \quad \text{if } S_T \geq X$$

$$= 0 \quad \text{if } S_T < X.$$

Let

$$y(u,0) = X\left(e^{u(\sigma^2/2)/(r-\sigma^2/2)} - 1\right) \quad \text{if } u \geq 0$$

$$= 0 \quad \text{if } u < 0.$$

At $\tau = 0$, u will equal $(2/\sigma^2)(r - \sigma^2/2)\ln(S_T/X)$. So at $\tau = 0$,

$$\begin{aligned} y(u,0|\tau = 0) &= X(e^{\ln(S_T/X)} - 1) = S_T - X & \text{if } u \geq 0 \\ &= 0 & \text{if } u < 0. \end{aligned}$$

Thus $y(u,0|\tau = 0)$ is consistent with $c(S_T,0)$. A function $f(x)$ that meets this condition is

$$\begin{aligned} f(u) &= X(e^{u(\sigma^2/2)/(r-\sigma^2/2)} - 1) & \text{if } u \geq 0 \\ &= 0 & \text{if } u < 0. \end{aligned}$$

The solution is, thus,

$$y = \frac{1}{\sqrt{\pi}} \int_{-u/2\sqrt{q}}^{\infty} X \left[e^{(u+2\eta\sqrt{q})(\sigma^2/2)/(r-\sigma^2/2)} - 1 \right] e^{-\eta^2} d\eta.$$

The lower limit was changed from $-\infty$ to $-u/2\sqrt{q}$ because we require the integrand to be non-negative to get a positive value of y . Thus, the first exponential function must be greater than or equal to one. So the exponent must be restricted to $(u + 2\eta\sqrt{q}) \geq 0$ meaning that $\eta \geq -u/2\sqrt{q}$.

Now let $\eta = a\sqrt{2}/2$. We will have $-\eta^2 = -a^2/2$, $d\eta = da/\sqrt{2}$, and the lower limit of integration will change to $-u/\sqrt{2q}$. Then the equation will become

$$y = \int_{-u/\sqrt{2q}}^{\infty} \frac{1}{\sqrt{\pi}} X \left[e^{(u+a\sqrt{2q})(\sigma^2/2)/(r-\sigma^2/2)} - 1 \right] e^{-a^2/2} da.$$

Now note that $-u/\sqrt{2q} = (1/\sigma) [\ln(S/X) - (r - \sigma^2/2)(-\tau)] / \sqrt{\tau} = -d_2$. Next we transform the above equation back into the equation for c_t by multiplying by $e^{-r\tau}$ and then we separate the integrals:

$$\begin{aligned} c &= \int_{-d_2}^{\infty} \frac{1}{\sqrt{2\pi}} X e^{-r\tau} \left[e^{(u+2a\sqrt{q})(\sigma^2/2)/(r-\sigma^2/2)} - 1 \right] e^{-a^2/2} da \\ &= \int_{-d_2}^{\infty} \frac{1}{\sqrt{2\pi}} X e^{-r\tau} e^{\left[(u+2a\sqrt{q})(\sigma^2/2)/(r-\sigma^2/2) \right] - a^2/2} da - X e^{-r\tau} \int \frac{1}{\sqrt{2\pi}} e^{-a^2/2} da. \end{aligned}$$

The first term can be simplified by using the definitions of u and q . After rearranging those definitions, we have

$$\begin{aligned} u(2/\sigma^2)(r - \sigma^2/2) &= [\ln(S/X) - (r - \sigma^2/2)(-\tau)] \\ a\sqrt{2q}(1/\sigma^2)(r - \sigma^2/2) &= -a\sigma\sqrt{\tau}. \end{aligned}$$

Thus, the first term in our solution is

$$\begin{aligned}
& X e^{-r\tau} e^{\ln(S/X)} \int_{-d_2}^{\infty} \frac{1}{\sqrt{2\pi}} e^{r\tau} e^{[-a^2/2 + 2a\sigma\sqrt{\tau} - \sigma^2\tau]} da \\
&= S \int_{-d_2}^{\infty} e^{-(1/2)[a^2 - 2a\sigma\sqrt{\tau} + \sigma^2\tau]} da.
\end{aligned}$$

The integrand is $e^{-(1/2)(a - \sigma\sqrt{\tau})^2}$. Now define $b = a - \sigma\sqrt{\tau}$ so $db = da$ and the lower limit of integration becomes $-d_2 - \sigma\sqrt{\tau}$. Let $d_1 = d_2 + \sigma\sqrt{\tau}$ so we have

$$S \int_{-d_1}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-b^2/2} db,$$

which is simply $S[1 - N(-d_1)] = SN(d_1)$.

The second term in the solution is simply $-Xe^{-r\tau}[1 - N(-d_2)] = -Xe^{-r\tau}N(d_2)$. Thus,

$$c = SN(d_1) - Xe^{-r\tau}N(d_2),$$

which is the Black-Scholes formula.

An alternative route to the solution uses the Laplace transform on the partial differential equation. This converts it to an ordinary differential equation, for which standard solution techniques can be applied.

Appendix B. Partial Derivatives of the Black-Scholes Put Option Pricing Model

$$\frac{\partial p}{\partial S} = N(d_1 - 1) \leq 0$$

$$\frac{\partial^2 p}{\partial S^2} = \frac{1}{S\sigma\sqrt{\tau}} \frac{\partial N(d_1)}{\partial d_1} \geq 0$$

$$\frac{\partial p}{\partial \tau} = \frac{S}{2\sqrt{\tau}} \frac{\partial N(d_1)}{\partial d_1} - rXe^{-r\tau}[1 - N(d_2)] \geq 0$$

$$\frac{\partial p}{\partial \sigma} = S\sqrt{\tau} \frac{\partial N(d_1)}{\partial d_1} \geq 0$$

$$\frac{\partial p}{\partial r} = -\tau X e^{-rt} [1 - N(d_2)] \leq 0$$

$$\frac{\partial p}{\partial X} = e^{-rt} [1 - N(d_2)] \geq 0$$

References

- Bachelier, L. "Theory of Speculation." English translation by A. J. Boness, *The Random Character of Stock Market Prices*, ed. P. Cootner. Cambridge: Massachusetts: The M.I.T. Press (1964), 17-78.
- Beck, T. M. "Black-Scholes Revisited: Some Important Details." *The Financial Review* 28 (February, 1993), 77-90.
- Black, F. "How to Use the Holes in Black-Scholes." *Journal of Applied Corporate Finance* 1 (Winter, 1989), 67-73.
- Black, F. "How We Came Up with the Option Formula." *Journal of Portfolio Management* 15 (Winter, 1989), 4-8.
- Black, F. and M. Scholes. "The Pricing of Options and Corporate Liabilities." *Journal of Political Economy* 81 (May-June, 1973), 637-654.
- Black, F. and M. Scholes. "The Valuation of Option Contracts and a Test of Market Efficiency." *The Journal of Finance* 27 (May, 1972), 399-417.
- Boness, A. J. "Elements of a Theory of Stock-Option Value." *Journal of Political Economy* 72 (April, 1964), 163-175.
- Chance, D. M. "Translating the Greek: An Alternative Interpretation of Call Option Derivatives." *Financial Analysts Journal* 50 (July-August, 1994), 43-49.
- Churchill, R. V. *Fourier Series and Boundary Problems*. New York: McGraw-Hill (1963).
- Conine, T. E., Jr. and M. Tamarkin. "A Pedagogic Note on the Derivation of the Comparative Statics of the Option Pricing Model." *The Financial Review* (1984), 397-400.
- Dothan, M. U. *Prices in Financial Markets*. New York: Oxford University Press (1990), Ch.9.
- Duffie, D. *Dynamic Asset Pricing Theory*. Princeton, New Jersey: Princeton University Press (1992), Ch. 5, 7.
- Duffie, D. *Security Markets: Stochastic Models*. Boston: Academic Press (1988), Ch. 22.

- Garven, J. R. "A Pedagogic Note on the Derivation of the Black-Scholes Option Pricing Formula." *The Financial Review* 21 (1986), 337-344.
- Gibson, R. *Option Valuation: Analyzing and Pricing Standardized Option Contracts*. New York: McGraw-Hill (1991), Chs. 4, 5.
- Hull, J. *Options, Futures and Other Derivatives*. Upper Saddle River, New Jersey: Prentice-Hall (2003), Ch. 12.
- Ingersoll, J. E. *Theory of Financial Decision Making*. Totowa, New Jersey: Rowman and Littlefield (1987), Chs. 14, 17.
- Jarrow, R. and A. Rudd. *Option Pricing*. Homewood, Illinois: R. D. Irwin (1983), Ch. 7-9.
- Kutner, G. W. "Black-Scholes Revisited: Some Important Details." *The Financial Review* 23 (February, 1988), 95-104.
- Merton, R. C., Jr. "Theory of Rational Option Pricing." *Bell Journal of Economics and Management Science* 4 (Spring, 1973), 141-183.
- Ritchken, P. *Derivative Markets: Theory, Strategy, and Applications*. New York: HarperCollins (1996), Ch. 9.
- Rubinstein, M. E. "The Valuation of Uncertain Income Streams and the Pricing of Options." *Bell Journal of Economics* 7 (Autumn, 1976), 407-425.
- Samuelson, P. A. "Rational Theory of Warrant Pricing." *Industrial Management Review* 6 (Spring, 1967), 13-31.
- Shimko, D. C. *Finance in Continuous Time*. Miami: Kolb Publishing Co. (1992), Ch. 3.
- Smith, C. W., Jr. "Option Pricing: A Review." *Journal of Financial Economics* 3 (January-March, 1976), 3-51.
- Sprenkle, C. M. "Warrant Prices as Indicators of Expectations and Preferences." *Yale Economic Essays* 1 (1962), 172-231.
- Stoll, H. R. and R. Whaley. *Futures and Options: Theory and Applications*. Cincinnati: South-Western Publishing (1993), Ch. 11.
- Wilmott, P., S. Howison, and J. DeWynne. *The Mathematics of Financial Derivatives*. Cambridge, U.K.: Cambridge University Press (1995), Chs. 3, 4, 5.