

Functional Coefficient Estimation with Both Categorical and Continuous Data*

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Abstract

We propose a local linear functional coefficient estimator that admits a mix of discrete and continuous data for stationary time series. Under weak conditions our estimator is asymptotically normally distributed. A small set of simulation studies is carried out to illustrate the finite sample performance of our estimator. As an application, we estimate a wage determination function that explicitly allows the return to education to depend on other variables. We find evidence of the complex interacting patterns among the regressors in the wage equation, such as increasing returns to education when experience is very low, high return to education for workers with several years of experience, and diminishing returns to education when experience is high. Compared with the commonly used parametric and semi-parametric methods, our estimator performs better in both goodness-of-fit and in yielding economically interesting interpretation.

Key words: Discrete variables, Functional coefficient estimation, Local linear estimation, Cross-validation.

JEL Classification: C13, C14.

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1 Introduction

In this paper we extend the work of Racine and Li (2004) to estimating functional coefficient models with both continuous and categorical data:

$$Y = \sum_{j=1}^d a_j(U) X_j + \varepsilon, \quad (1.1)$$

where ε is the disturbance term, X_j is a scalar random variable, U is a $(p+q) \times 1$ random vector, and $a_j(\cdot)$, $j = 1, \dots, d$, are unknown smooth functions. As Cai, Fan and Yao (2000) remark, the idea for this kind of model is not new, but the potential of this modeling techniques had not been fully explored until the seminal work of Cleveland et al. (1992), Chen and Tsay (1993), and Hastie and Tibshirani (1993), in which nonparametric techniques were proposed to estimate the unknown functions $a_j(\cdot)$. An important feature of these early works is to assume that the random variable U is continuous, which limits the model's potential applications.

Drawing upon the work of Aitchison and Aitken (1976), Racine and Li (2004) propose a novel approach to estimate nonparametric regression mean functions with both categorical and continuous data in the iid setup. They apply their new estimation method to some publicly available data and demonstrate the superb performance of their estimators in comparison with some traditional ones.

In this paper, we consider extending the work of Racine and Li (2004) to the estimation of the functional coefficient model (1.1) when U contains both continuous and categorical variables. This is important since categorical variables may be present in the functional coefficients. For example, in the study of the output functions for individual firms, firms that belong to different industries may exhibit different output elasticities with respect to labor and capital. So we should allow the categorical variable 'industry' to enter U . We will demonstrate that this modelling strategy outperforms the traditional dummy-variable approach widely used in the literature.

Another distinguishing feature of our approach is that we allow for weak data dependence. One of the key applications of nonparametric function estimation is the construction of prediction intervals for stationary time series. The iid setup of Racine and Li (2004) cannot meet this purpose.

To demonstrate the usefulness of our proposed estimator in empirical applications, we estimate a wage determination equation based on recent CPS data. While in the literature of labor economics, the return to education has already been extensively investigated from various aspects, in this paper, we explicitly allow the return to education to be dependent on other variables, both continuous and discrete, including experience, gender, age, industry and so forth. Our findings are clearly against the parametric functional form assumption of the most widely used linear separable Mincerian equation, and the return to education does vary substantially with the other regressors. Therefore, our model can help to uncover economically interesting interacting effects among the regressors, and so should have high potential for applications.

The paper is structured as follows. In Section 2 we introduce our functional coefficient estimators and their asymptotic properties. We conduct a small set of Monte Carlo studies to check the relative performance of the proposed estimator in Section 3. Section 4 provides empirical data analysis. Final

remarks are contained in Section 5. All technical details are relegated to the Appendix.

2 Functional Coefficient Estimation with Mixed Data

2.1 Local linear estimator

In this paper, we study estimation of model (1.1) when U is comprised of a mix of discrete and continuous variables. Let $\{(Y_i, X_i, U_i), i = 1, 2, \dots, n\}$ be jointly strictly stationary processes, where (Y_i, X_i, U_i) has the same distribution as (Y, X, U) . Let $U_i = (U_i^c, U_i^d)$, where U_i^c and U_i^d denote a $p \times 1$ vector of continuous regressors and a $q \times 1$ vector of discrete regressors, respectively. Like Racine and Li (2004), we will use U_{it}^d to denote the t th component of U_i^d , and assume that U_{it}^d can take $c_t \geq 2$ different values, i.e., $U_{it}^d \in \{0, 1, \dots, c_t - 1\}$ for $t = 1, \dots, q$. Denote $u = (u^c, u^d) \in \mathbb{R}^p \times \mathbb{R}^q$. We use $f_u(u) = f(u^c, u^d)$ to denote the joint density function of (U_i^c, U_i^d) and $\mathcal{D} = \prod_{t=1}^q \{0, 1, \dots, c_t - 1\}$ to denote the range assumed by U_i^d . With a little abuse of notation, we also use $\{(Y_i, X_i, U_i), i = 1, \dots, n\}$ to denote the data.

To define the kernel weight function, we focus on the case for which there is no natural ordering in U_i^d . Define

$$l(U_{it}^d, u_t^d, \lambda_t) = \begin{cases} 1 & \text{if } U_{it}^d = u_t^d, \\ \lambda_t & \text{if } U_{it}^d \neq u_t^d, \end{cases} \quad (2.1)$$

where λ_t is a bandwidth that lies on the interval $[0, 1]$. Clearly, when $\lambda_t = 0$, $l(U_{it}^d, u_t^d, 0)$ becomes an indicator function, and $\lambda_t = 1$, $l(U_{it}^d, u_t^d, 1)$ becomes a uniform weight function. We define the product kernel for the discrete random variables by

$$L(U_i^d, u^d, \lambda) = \prod_{t=1}^q l(U_{it}^d, u_t^d, \lambda_t). \quad (2.2)$$

For the continuous random variables, we use $w(\cdot)$ to denote a univariate kernel function and define the product kernel function by $W_{h,iu} = \prod_{t=1}^p w((U_{it}^c - u_t^c)/h_t)$, where $h = (h_1, \dots, h_p)$ denotes the smoothing parameters and U_{it}^c (u_t^c) is the t th component of U_i^c (u^c). We then define the kernel weight function K_{iu} by

$$K_{iu} = L_{\lambda,iu} W_{h,iu} \quad (2.3)$$

where $L_{\lambda,iu} = L(U_i^d, u^d, \lambda)$.

We now estimate the unknown functional coefficient functions in model (1.1) by using a local linear regression technique. Suppose that $a_j(\cdot)$ assumes a second order derivative. Denote by $\dot{a}_j(u) = \partial a_j(u) / \partial u^c$ the $p \times 1$ first order derivative of $a_j(u)$ with respect to its continuous-valued argument u^c . Denote by $\ddot{a}_j(u) = \partial^2 a_j(u) / (\partial u^c \partial u^{c'})$ the $p \times p$ second order derivative matrix of $a_j(u)$ with respect to u^c . We use $a_{j,ss}(u)$ to denote the s th diagonal element of $\ddot{a}_j(u)$.

For any given u and \tilde{u} in a neighborhood of u , it follows from a first order Taylor expansion that

$$a_j(\tilde{u}) \approx a_j(u) + \dot{a}_j(u)'(\tilde{u}^c - u^c), \quad (2.4)$$

for u^c in a neighborhood of \tilde{u}^c and $\tilde{u}^d = u^d$. To estimate $\{a_j(u)\}$ (and $\{a'_j(u)\}$), we choose $\{a_j\}$ and $\{b_j\}$ to minimize

$$\sum_{i=1}^n \left[Y_i - \sum_{j=1}^d \{a_j + b'_j (U_i - u)\} X_{ij} \right]^2 K_{iu}. \quad (2.5)$$

Let $\{(\hat{a}_j, \hat{b}_j)\}$ be the local linear estimator. Then the local linear regression estimator for the functional coefficient is given by

$$\hat{a}_j(u) = \hat{a}_j, \quad j = 1, \dots, d. \quad (2.6)$$

The local linear regression estimator for the functional coefficient can be easily obtained. To do so, let $\mathbf{e}_{j,d(p+1)}$ be the $d(1+p) \times 1$ unit vector of with 1 at the j th position and 0 elsewhere. Let $\tilde{\mathbf{X}}$ denote an $n \times d(1+p)$ matrix with

$$\tilde{X}_i = (X'_i, X'_i \otimes (U_i - u)')$$

as its i th row. Let $\mathbf{Y} = (Y_1, \dots, Y_n)'$. Set $\mathbf{W} = \text{diag}\{K_{1u}, \dots, K_{nu}\}$. Then (2.5) can be written as

$$(\mathbf{Y} - \tilde{\mathbf{X}}\theta)' \mathbf{W} (\mathbf{Y} - \tilde{\mathbf{X}}\theta),$$

where $\theta = (a_1, \dots, a_d, b'_1, \dots, b'_d)'$. So the local linear estimator is simply

$$\hat{\theta} = (\tilde{\mathbf{X}}' \mathbf{W} \tilde{\mathbf{X}})^{-1} \tilde{\mathbf{X}}' \mathbf{W} \mathbf{Y}, \quad (2.7)$$

which entails that

$$\hat{a}_j(u) = \hat{a}_j = \mathbf{e}'_{j,d(1+p)} \hat{\theta}, \quad j = 1, \dots, d. \quad (2.8)$$

We will study the asymptotic properties of $\hat{\theta}$.

2.2 Assumptions

To facilitate the presentation, let $\Omega(u) = E(X_i X'_i | U_i = u)$, $\sigma^2(u, x) = E[\varepsilon_i^2 | U_i = u, X_i = x]$, $\Omega^*(u) = E[X_i X'_i \sigma^2(U_i, X_i) | U_i = u]$. Let $f(u, x)$ denote the joint density of (U_i, X_i) and $f_u(u)$ be the marginal density of U_i . Also, let $f_{u|x}(u|x)$ be the conditional density of U_i given $X_i = x$. Let $f_i(u, \tilde{u}|x, \tilde{x})$ be the conditional density of (U_1, U_i) given $(X_1, X_i) = (x, \tilde{x})$.

We now list the assumptions that will be used to establish the asymptotic distribution of our estimator.

Assumption

A1. (i) The process $\{(Y_i, U_i, X_i), i \geq 1\}$ is a strictly stationary α -mixing process with coefficients $\alpha(n)$ satisfying $\sum_{j \geq 1} j^c [\alpha(j)]^{\gamma/(2+\gamma)} < \infty$ for some $\gamma > 0$ and $c > \gamma/(2+\gamma)$.

(ii) $f_{u|x}(u|x) \leq M < \infty$ and $f_i(u, \tilde{u}|x, \tilde{x}) \leq M < \infty$ for all $i \geq 2$ and $u, \tilde{u}, x, \tilde{x}$.

(iii) $\Omega^*(u)$ and $\Omega(u)$ are positive definite.

(iv) The functions $f_u(\cdot, u^d)$, $\sigma^2(\cdot, u^d, x)$, $\Omega(\cdot, u^d)$, and $\Omega^*(\cdot, u^d)$ are continuous for all $u^d \in \mathcal{D}$, and $f_u(u) > 0$.

- (v) $a_j(\cdot, u^d)$ has continuous second derivatives for all $u^d \in \mathcal{D}$.
- (vi) $E \|X\|^{2(2+\gamma)} < \infty$, where $\|\cdot\|$ is the Euclidean norm and γ is given in (i).
- (vii) $E [Y_1^2 + Y_i^2 | (U_1, X_1) = (u, x); (U_i, X_i) = (\tilde{u}, \tilde{x})] \leq M < \infty$.
- (viii) There exists $\delta > 2 + \gamma$ such that $E [Y_1^\delta | (U_1, X_1) = (\tilde{u}, x)] \leq M < \infty$ for all $x \in \mathbb{R}^d$ and all \tilde{u} in the neighborhood of u . $\alpha(j) = O(j^{-\kappa})$, where $\kappa \geq (2 + \gamma)\delta / \{2(\delta - 2 - \gamma)\}$.
- (ix) There exists a sequence of positive integers s_n such that $s_n \rightarrow \infty$, $s_n = o((nh_1 \dots h_p)^{1/2})$, and $n^{1/2} (h_1 \dots h_p)^{-1/2} \alpha(s_n) \rightarrow 0$.

A2. The kernel function $w(\cdot)$ is a density function that is symmetric, bounded, and compactly supported.

A3. As $n \rightarrow \infty$, the bandwidth sequences $h_s \rightarrow 0$, for $s = 1, \dots, p$, $\lambda_s \rightarrow 0$, for $s = 1, \dots, q$, and (i) $nh_1 \dots h_p \rightarrow \infty$, (ii) $(nh_1 \dots h_p)^{1/2} (\|h\|^2 + \|\lambda\|) = O(1)$.

Assumptions A1-A2 are similar to Conditions A and B in Cai, Fan and Yao (2000) except that we consider mixed regressors. Assumption A1(i) is standard in the nonparametric regression for time series. See, for example, Cai, Fan and Yao (2000), and Cai and Ould-Saïd (2003). It is satisfied by many well-known processes such as linear stationary ARMA processes and a large class of processes implied by numerous nonlinear models, including bilinear, nonlinear autoregressive (NLAR), and ARCH-type models (see Fan and Li, 1999). As Hall et al. (1999) and Cai and Ould-Saïd (2003) remark, the requirement in Assumption A2 that $w(\cdot)$ is compactly supported can be removed at the cost of lengthier arguments used in the proofs, and in particular, Gaussian kernel is allowed.

Assumption A3 is standard for nonparametric regression with mixed data (see Li and Racine, 2005).

2.3 Asymptotic theory for the local linear estimator

To introduce our main results, let $\mu_{s,t} = \int_{\mathbb{R}} v^s w(v)^t dv$, $s, t = 0, 1, 2$. Define two $d(1+p) \times d(1+p)$ diagonal matrices $S = S(u)$ and $\Gamma = \Gamma(u)$ by

$$S = f_u(u) \begin{pmatrix} \Omega(u) & 0'_{dp \times d} \\ 0_{dp \times d} & \mu_{2,1} \Omega(u) \otimes I_p \end{pmatrix}, \quad \Gamma = f_u(u) \begin{pmatrix} \mu_{0,2}^p \Omega^*(u) & 0'_{dp \times d} \\ 0_{dp \times d} & \mu_{2,2} \Omega^*(u) \otimes I_p \end{pmatrix},$$

where $0_{l \times k}$ is an $l \times k$ matrix of zeros, I_p is the $p \times p$ identity matrix, and \otimes is the Kronecker product. For any $p \times 1$ vectors $c = (c_1, \dots, c_p)'$ and $d = (d_1, \dots, d_p)'$, let $c \odot d \equiv (c_1 d_1, \dots, c_p d_p)'$.

To describe the leading bias term associated with the discrete random variables, we define an indicator function $I_s(\cdot, \cdot)$ by

$$I_s(u^d, \tilde{u}^d) = 1(u_s^d \neq \tilde{u}_s^d) \prod_{t \neq s}^q 1(u_t^d = \tilde{u}_t^d).$$

That is, $I_s(u^d, \tilde{u}^d)$ is one if and only if u^d and \tilde{u}^d differ only in the s th component and is zero otherwise.

Let

$$b(h, \lambda) = H \left\{ \begin{aligned} & \left(\begin{array}{c} \frac{1}{2} \mu_{2,1} f_u(u) \Omega(u) A \\ 0_{dp \times 1} \end{array} \right) \\ & + \sum_{s=1}^q \lambda_s I_s(u^d, \tilde{u}^d) f_u(u^c, \tilde{u}^d) \left(\begin{array}{c} \Omega(u^c, \tilde{u}^d) (\mathbf{a}(u^c, \tilde{u}^d) - \mathbf{a}(u)) \\ -\mu_{2,1} (\Omega(u^c, \tilde{u}^d) \otimes I_p) \mathbf{b}(u) \end{array} \right) \end{aligned} \right\}, \quad (2.9)$$

where $H = \sqrt{nh_1 \dots h_p}$, $A = (\sum_{s=1}^p h_s^2 a_{1,ss}(u), \dots, \sum_{s=1}^p h_s^2 a_{d,ss}(u))'$, $\mathbf{a}(u) = (a_1(u), \dots, a_d(u))'$, and $\mathbf{b}(u) = (\dot{a}_1(u)', \dots, \dot{a}_d(u)')$. Define $B_{j,1s}(u) = (\mu_{2,1}/2) a_{j,ss}(u)$ and

$$B_{j,2s}(u) = \mu_2 f_u(u)^{-1} \left\{ \sum_{\tilde{u}^d \in \mathcal{D}} I_s(u^d, \tilde{u}^d) f(u^c, \tilde{u}^d) (a_j(u^c, \tilde{u}^d) - a_j(u)) \right\}.$$

Now we state our main theorem.

Theorem 2.1 *Assume that Assumptions A1-A3 hold. Then*

$$HH_1 (\hat{\theta} - \theta) - S^{-1} b(h, \lambda) \xrightarrow{d} N(0, S^{-1} \Gamma S^{-1}).$$

where $H_1 = \text{diag}(1, \dots, 1, h', \dots, h')$ is a $d(p+1) \times 1$ diagonal matrix with d diagonal elements of 1 and d diagonal elements of h . In particular, for $j = 1, \dots, d$,

$$\begin{aligned} & \sqrt{nh_1 \dots h_p} \left(\hat{a}_j - a_j(u) - \sum_{s=1}^p h_s^2 B_{j,1s}(u) - \sum_{s=1}^q \lambda_s B_{j,2s}(u) \right) \\ & \xrightarrow{d} N \left(0, \frac{\mu_{0,2}^p \mathbf{e}'_{j,d} \Omega^{-1}(u) \Omega^*(u) \Omega^{-1}(u) \mathbf{e}_{j,d}}{f_u(u)} \right). \end{aligned}$$

Remark. Noting that S and Γ are both block diagonal matrices, we have asymptotic independence between the estimator for $\mathbf{a}(u)$ and that for $\mathbf{b}(u)$. Under Assumption A3, the asymptotic bias of \hat{a}_j is comprised of two components, $\sum_{s=1}^p h_s^2 B_{j,1s}(u)$ and $\sum_{s=1}^q \lambda_s B_{j,2s}(u)$, which are associated with the continuous and discrete variables, respectively.

2.4 Selection of smoothing parameters

In this subsection we focus on how to choose the smoothing parameters for \hat{a}_j . It is well known that the choice of smoothing parameters is crucial in nonparametric kernel estimation.

Theorem 2.1 implies that the leading term for the mean squared error (MSE) of \hat{a}_j is

$$MSE(\hat{a}_j) = \left[\sum_{s=1}^p h_s^2 B_{j,1s}(u) + \sum_{s=1}^q \lambda_s B_{j,2s}(u) \right]^2 + \frac{1}{nh_1 \dots h_p} \frac{\mu_{0,2}^p \mathbf{e}'_{j,d} \Omega^{-1}(u) \Omega^*(u) \Omega^{-1}(u) \mathbf{e}_{j,d}}{f_u(u)}.$$

By symmetry, all h_j should have the same order and all λ_s should also have the same order but with $\lambda_j \sim h_j^2$. By an argument similar to Li and Racine (2005), it is easy to obtain the optimal rate of

bandwidth in terms of minimizing a weighted integrated version of $MSE(\hat{a}_j)$. To be concrete, we should choose

$$h_j \sim n^{-1/(4+p)} \text{ and } \lambda_j \sim n^{-2/(4+p)}.$$

Nevertheless, the exact formula for the optimal smoothing parameters is difficult to obtain except for the simplest cases (e.g., $p = 1$ and $q \leq 1$). This also suggests that it is infeasible to use the plug-in bandwidth in applied setting since the plug-in method would first require the formula for each smoothing parameter and then pilot estimates for some unknown functions in the formula.

In practice, we propose to use least squares cross-validation to choose the smoothing parameters. We choose (h, λ) to minimize the following least squares cross validation criterion function

$$CV(h, \lambda) = \frac{1}{n} \sum_{i=1}^n \left(Y_i - \sum_{j=1}^d \hat{a}_j^{(-i)}(U_i) X_{ij} \right)^2,$$

where $\hat{a}_j^{(-i)}(U_i)$ is the leave-one-out functional coefficient estimator of $a_j(U_i)$. Let $(\hat{h}, \hat{\lambda})$ denote the solution to the above problem. It will be used in the following study.

3 Monte Carlo Simulations

We now conduct Monte Carlo experiment to illustrate the finite sample performance of our nonparametric functional coefficient estimators with mixed data. In addition to the proposed estimator, we also include several other parametric and nonparametric estimators.

The first data generating process (DGP) we consider is given by

$$Y_i = 0.1 (U_{i1}^2 + U_{i2} + U_{i3}) + 0.1 (U_{i1}U_{i2} + U_{i3}) X_{i1} + 0.15 (U_{i1}U_{i2} + U_{i3}) X_{i2} + \varepsilon_i,$$

where $X_{ij} \sim \text{Uniform}(0, 4)$ ($j = 1, 2$), $U_{i1} \sim \text{Uniform}(0, 4)$, $U_{ij} \in \{0, 1, \dots, 5\}$ with $P(U_{ij} = l) = 1/6$ for $l = 0, 1, \dots, 5$ and $j = 2, 3$, and $\varepsilon_i \sim N(0, 1)$. Furthermore, X_{ij} , U_{ij} , and ε_i are iid and mutually independent.

We consider two nonparametric estimators and three parametric estimators for the conditional mean function $m(x, u) = E(Y_i | X_i = x, U_i = u)$. We first obtain our nonparametric functional coefficient estimator (NP) with mixed data where the smoothing parameters (h, λ) are chosen by the least squares cross-validation. Then we obtain the nonparametric frequency estimator (NP-FREQ) with mixed data by using the cross-validated h and setting $\lambda = 0$ (see Li and Racine, 2007, Ch 3). It is expected that the smaller the ratio of the sample size to the number of ‘‘cells’’, the worse the nonparametric frequency approach relative to our proposed kernel approach.

For the parametric estimation, we consider in practice what an applied econometrician would do when he or she confronts the data $\{Y_i, X_i, U_i\}_{i=1}^n$ and have a strong belief that all the variables in X_i and U_i can affect the dependent variable Y_i . In the first parametric model, we ignore the potential interaction between regressors and estimate a linear model without any interaction (LIN) by regressing Y_i on X_i , U_{i1} , and the dummy variables created from the two categorical variables U_{i2} and U_{i3} . In

Table 1: Comparison of finite sample performance of various estimators (DGP1)

n	Model	Mean	Median	Sd dev	IQR
100	NP	0.731	0.713	0.138	0.176
	NP-FREQ	0.995	0.993	0.158	0.185
	LIN	2.395	2.336	0.553	0.720
	LIN-INT1	1.694	1.637	0.359	0.458
	LIN-INT2	1.088	1.072	0.211	0.283
200	NP	0.525	0.524	0.071	0.085
	NP-FREQ	0.884	0.886	0.100	0.120
	LIN	2.473	2.461	0.380	0.533
	LIN-INT1	1.777	1.767	0.250	0.362
	LIN-INT2	1.142	1.138	0.153	0.229
400	NP	0.371	0.368	0.052	0.057
	NP-FREQ	0.558	0.547	0.065	0.076
	LIN	2.487	2.471	0.376	0.340
	LIN-INT1	1.780	1.785	0.196	0.263
	LIN-INT2	1.134	1.132	0.110	0.162

the second parametric model, we take into account potential interaction between X_i and U_{1i} , and estimate a linear model with interaction (LIN-INT1) by adding the interaction terms between X_i and U_{1i} into the LIN model. In the third parametric model, we also consider the interaction between X_i and (U_{2i}, U_{3i}) , so we estimate a linear model with interaction (LIN-INT2) by adding the interaction terms between X_i and (U_{1i}, U_{2i}, U_{3i}) into the LIN-INT2 model. We expect LIN-INT2 outperforms LIN-INT1, which in turn outperforms LIN in terms of mean squared errors.

For performance measure, we compute the in-sample mean-square error (MSE) using

$$MSE = \frac{1}{n} \sum_{i=1}^n \{m(X_i, U_i) - \hat{m}(X_i, U_i)\}^2,$$

where $\hat{m}(X_i, U_i)$ is the estimator for the conditional mean $m(X_i, U_i)$ using different methods introduced earlier. We report the mean, median, standard error, and interquartile range of MSE over 500 Monte Carlo replications. We set the sizes for the estimation samples to be $n = 100, 200, \text{ and } 400$.

Table 1 reports the results from all five regression models. From Table 1 we observe that our proposed nonparametric functional coefficient estimator dominates both the conventional nonparametric frequency estimator and the three parametric models in terms of MSE.

We now consider a second DGP which allows for data dependence between observations. The data are generated from the following DGP

$$Y_i = U_{i1} (U_{i1} + U_{i2} + U_{i3}) + U_{i1} (U_{i1} + U_{i2} + U_{i3}) X_i + \varepsilon_i,$$

Table 2: Comparison of finite sample performance of various estimators (DGP2)

n	Model	Mean	Median	Sd dev	IQR
100	NP	0.396	0.377	0.138	0.156
	NP-FREQ	0.495	0.459	0.171	0.249
	LIN	2.544	2.373	0.895	1.037
	LIN-INT1	1.946	1.848	0.633	0.798
	LIN-INT2	0.391	0.365	0.146	0.157
200	NP	0.245	0.217	0.113	0.125
	NP-FREQ	0.286	0.247	0.123	0.170
	LIN	2.634	2.552	0.698	0.985
	LIN-INT1	2.067	1.973	0.515	0.635
	LIN-INT2	0.389	0.376	0.112	0.145
400	NP	0.144	0.123	0.057	0.078
	NP-FREQ	0.156	0.130	0.066	0.091
	LIN	2.675	2.628	0.449	0.598
	LIN-INT1	2.087	2.082	0.336	0.466
	LIN-INT2	0.385	0.379	0.076	0.097

where

$$\begin{aligned} X_i &= 0.5X_{i-1} + e_{i1}, \\ U_{i1} &= 0.5U_{i-1,1} + e_{i2}, \end{aligned}$$

$\varepsilon_i \sim N(0, 1)$, $e_{ij} \sim N(0, 1)$ ($j = 1, 2$), $U_{ij} \in \{-1, 0, 1\}$ with $P(U_{ij} = l) = 1/3$ for $l = -1, 0, 1$ and $j = 2, 3$. Furthermore, e_{ij} ($j = 1, 2$), U_{i2} , U_{i3} , and ε_i are iid and mutually independent.

Like the case for DGP1, we also consider two nonparametric estimators and three parametric estimators for the conditional mean function $m(x, u) = E(Y_i | X_i = x, U_i = u)$. We denote the corresponding regression models as NP, NP-FREQ, LIN, LIN-INT1, and LIN-INT2, respectively. We again consider the performance measure in terms of MSE . We report the mean, median, standard error, and interquartile range of MSE over 500 Monte Carlo replications. We set the sizes for the estimation samples to be $n = 100, 200$, and 400 . The results are reported in Table 2. From Table 2 we observe that our proposed nonparametric functional coefficient estimator dominates both the conventional nonparametric frequency estimator and the three parametric models in terms of MSE .

4 An Empirical Application: Estimating the Wage Equation

In this section, we apply our functional coefficient model to estimate a wage equation embedded in the framework of Mincer's (1974) human capital earning function. The basic Mincer wage function takes the form:

$$\log Y = \beta_0 + \beta_1 S + \beta_2 A + \beta_3 A^2 + \epsilon, \tag{4.1}$$

where Y is some measure of individual earnings, S is years of schooling and A is age or work experience. In spite of its simplicity, Mincer Equation captures the reality remarkably well (Card, 1999), and has been firmly established as a benchmark in labor economics. Concerning its specification, several extensions have been made to allow more general parametric functional forms (see Murphy and Welch, 1990). Further, a nonparametric analysis has been done in Ullah (1985) and Zheng (2000). And in practice, other control variables, such as indicators of gender, race, occupation, or marital status are routinely included in the wage equation when they are available. Nevertheless, the additive separability assumption of the standard Mincer equation may be too stringent. For instance, it ignores the possibility that higher education results in more return to seniority. Also, it is often of keen economic and policy interest to investigate the differentials among different gender and race groups, where the return to education or experience may differ substantially. Therefore, we intend to estimate the functional coefficient model of the following form:

$$\log Y = a_1(U) + a_2(U)S + \epsilon, \quad (4.2)$$

where Y and S are as defined above, and U is a vector of mixed variables including one continuous variable – age or work experience, and six categorical variables for gender, race, marital status, veteran status, industry, and geographic location. The specification of (4.2) enables us to both study the direct effects of variables in U flexibly, and investigate whether and how they influence the return to education. Some past literature has already suggested nonlinear relationship between seniority and wage beyond a quadratic form (Murphy and Welch, 1990, Ullah, 1985, Zheng, 2000), as well as the fact that rising return to education from the 1980s is more drastic in the younger cohorts than in the older ones (Card and Lemieux, 2001).

Our model is also suitable for analyzing the gender and racial wage differentials. In the study of discrimination, it is common practice to estimate a “gender/racial wage gap” or estimate wage equation in separate samples. (For a survey of race and gender in the labor market, see Altonji and Blank, 1999.) Here the limitation of application of the traditional nonparametric method is the fact that indicators for gender and race are discrete, a problem overcome in our model. Also, compared with estimating wage separately among gender-racial groups or the frequency approach, our approach utilizes the entire dataset, thus achieving efficiency gain. We can also explicitly address other supposedly complicated interaction effects between the variables of interest. Further, unlike a complete nonparametric specification, model (4.2) has the further advantage that it can be readily extended to instrument variable estimation (Cai et al., 2006), provided we have some reasonable instruments to correct the endogeneity in education. To keep our discussion focused, however, this aspect is not further explored in this paper.

The data utilized are drawn from March CPS data of the year 1990, 1995, 2000 and 2005. The earning variable is the weekly earning calculated from annual salary income divided by weeks of work, and deflated by the CPI (1982-1984=100). As usual, we exclude observations that are part-time workers, self-employed, over 65, under 18, or earn less than 50 dollars per week. All observations fall into 3 racial categories – white, Hispanic and otherwise, 4 geographic location categories – Northeast, Midwest, South and West, and 10 industrial categories. There are also three dichotomous variables

Table 3: Linear Wage Equation

Year	1990		1995		2000		2005	
	(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)
Education	0.098 ^a	(0.002)	0.107 ^a	(0.002)	0.105 ^a	(0.003)	0.107 ^a	(0.002)
Experience	0.029 ^a	(0.001)	0.036 ^a	(0.002)	0.029 ^a	(0.002)	0.031 ^a	(0.001)
Experience ²	-0.000 ^a	(0.000)	-0.001 ^a	(0.000)	-0.001 ^a	(0.000)	-0.001 ^a	(0.000)
Female	-0.309 ^a	(0.010)	-0.290 ^a	(0.010)	-0.279 ^a	(0.011)	-0.277 ^a	(0.008)
White	0.100 ^a	(0.013)	0.130 ^a	(0.013)	0.097 ^a	(0.013)	-0.098 ^a	(0.010)
Hispanic	0.034 ^c	(0.017)	0.040 ^c	(0.022)	0.033 ^c	(0.019)	0.034 ^b	(0.014)
Single	-0.087 ^a	(0.009)	-0.071 ^a	(0.010)	-0.097 ^a	(0.010)	-0.102 ^a	(0.008)
Veteran	-0.013	(0.013)	-0.049 ^a	(0.015)	-0.008	(0.016)	-0.031 ^b	(0.014)
Observations	12328		10834		10433		17466	
R^2	0.37		0.36		0.33		0.34	

Note: 1) Heteroskedasticity-robust standard errors in parentheses.

2) a , b and c stand for 1%, 5% and 10% significant levels, respectively.

3) 3 region indicators, 9 industry indicators and a constant in all specifications.

“Female”, “Veteran” and “Single”. Years of schooling are estimated by records of the highest educational degree attained and experience is approximated by *Age-Schooling-6*.

As a comparison, we also estimate a simple linear wage function, a linear wage function with interacting covariates, and a partially linear model. The results are reported in Table 3, Table 4, and Table 5 (see also Figure 1), respectively.

Results in Table 3 are in conformity with some stylized effects in labor economics, including stable return to schooling in the 1990s (Card and DiNardo, 2002; Beaudry and Green, 2004), concavity in return to experience, falling gender-wage gaps (Altonji and Blank, 1999), etc. Nevertheless, the inadequacy of a simple linear separable model is made clear in Table 4, since most of the interaction items of the covariates are significantly different from zero. And many of them are of important economic implications, such as the higher return to education for female and higher return to experience for the white. And the goodness-of-fit of the model after accounting for the interaction effects has also increased modestly.

Another extension of equation (4.1) is to consider the partially linear model: $\log Y = m(\textit{Schooling}, \textit{Experience}) + Z'\beta + \epsilon$, where Z is a set of dummy variables, and education and experience enter the model nonparametrically. Reported in Table 5, the partially linear model also performs better in goodness-of-fit, as expected. However, it is noteworthy that comparing with the simple linear model, accounting for the possibly complex function form of education and experience has also significantly changed the estimates of the coefficients for the other covariates. For instance, the effects of race have drastically dropped in magnitude as well as significance. The difference may be the result of biases induced by the misspecification in a parametric model, and thus indicates the needs for the more general functional form assumption.

Table 4: Linear Wage Equation with Interacted Regressors

Year	1990		1995		2000		2005	
	(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)
Education	0.133 ^a	(0.007)	0.146 ^a	(0.007)	0.134 ^a	(0.008)	0.151 ^a	(0.006)
Experience	0.059 ^a	(0.003)	0.071 ^a	(0.003)	0.049 ^a	(0.004)	0.053 ^a	(0.003)
Experience ²	-0.001 ^a	(0.000)	-0.001 ^a	(0.000)	-0.001 ^a	(0.000)	-0.001 ^a	(0.000)
Female	-0.349 ^a	(0.061)	-0.379 ^a	(0.069)	-0.526 ^a	(0.074)	-0.353 ^a	(0.059)
White	0.039	(0.089)	0.091	(0.091)	-0.077	(0.106)	0.025	(0.082)
Hispanic	0.496 ^a	(0.098)	0.551 ^a	(0.114)	0.455 ^a	(0.111)	0.607 ^a	(0.086)
Single	-0.132 ^a	(0.013)	-0.128 ^a	(0.014)	-0.137 ^a	(0.014)	-0.155 ^a	(0.012)
Veteran	-0.024 ^c	(0.014)	-0.056 ^a	(0.015)	-0.010 ^a	(0.017)	-0.027 ^c	(0.015)
Education×Experience	-0.002 ^a	(0.000)	-0.002 ^a	(0.000)	-0.001 ^a	(0.000)	-0.001 ^a	(0.000)
Education×Female	0.014 ^a	(0.004)	0.016 ^a	(0.004)	0.022 ^a	(0.005)	0.009 ^b	(0.004)
Education×White	0.009	(0.006)	0.007	(0.006)	0.010	(0.007)	0.006	(0.006)
Education×Hispanic	-0.034 ^a	(0.007)	-0.035 ^a	(0.008)	-0.039 ^a	(0.008)	-0.046 ^a	(0.006)
White×Female	-0.135 ^a	(0.025)	-0.123 ^a	(0.026)	-0.087 ^a	(0.026)	-0.098 ^a	(0.020)
Hispanic×Female	-0.017	(0.034)	-0.069	(0.043)	-0.035	(0.038)	0.012	(0.028)
Single×Female	0.114 ^a	(0.018)	0.141 ^a	(0.018)	0.105 ^a	(0.020)	0.135 ^a	(0.010)
Experience×Female	-0.005 ^a	(0.001)	-0.004 ^a	(0.001)	-0.002 ^a	(0.001)	-0.001 ^a	(0.001)
Experience×White	0.000	(0.001)	0.000	(0.001)	0.004 ^a	(0.001)	0.002 ^a	(0.001)
Experience×Hispanic	-0.003 ^c	(0.002)	-0.004 ^b	(0.002)	0.001	(0.002)	-0.000	(0.001)
Observations	12328		10834		10433		17466	
R^2	0.39		0.38		0.34		0.36	

Note: 1) Heteroskedasticity-robust standard errors in parentheses.

2) a,b and c stand for 1%, 5% and 10% significant levels, respectively.

3) 3 region indicators, 9 industry indicators and a constant in all specifications.

Table 5: Partially Linear Wage Equation

Year	1990		1995		2000		2005	
	(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)
Female	-0.280 ^a	(0.010)	-0.265 ^a	(0.011)	-0.259 ^a	(0.011)	-0.259 ^a	(0.008)
White	0.103 ^a	(0.012)	0.135 ^a	(0.013)	0.096 ^a	(0.013)	0.102 ^a	(0.010)
Hispanic	0.001	(0.017)	-0.001 ^a	(0.022)	-0.017	(0.019)	-0.007	(0.014)
Single	-0.077 ^a	(0.009)	-0.058 ^a	(0.010)	-0.082 ^a	(0.010)	-0.077 ^a	(0.008)
Veteran	0.024 ^a	(0.013)	-0.009	(0.015)	0.021	(0.016)	-0.001	(0.014)
Observations	12328		10834		10433		17446	
R^2	0.40		0.39		0.36		0.38	

Note: 1) Heteroskedasticity-robust standard errors in parentheses.

2) a , b and c stand for 1%, 5% and 10% significant levels, respectively.

3) 3 region indicators, 9 industry indicators and a constant in all specifications.

4) The estimate of $m(\textit{Schooling}, \textit{Experience})$ is plotted in Figure 1.

In all the above specifications, we use dummy variables to allow different intercepts for different regions and industries, and the majority of them have a significant estimated coefficient. The large number of categories makes it difficult to study their interaction effects with other regressors. In contrast, in the nonparametric framework of mixed regressors, only one categorical variable is necessary to describe such characteristic as industry or location. And this advantage has made our proposed model further suitable for the application.

For a comprehensive presentation of the regression results of model (4.2), we plot the wage-experience profiles of different cells defined by a discrete characteristic averaged over other categorical covariates. We use the second order Epanechnikov kernel in our nonparametric estimation: $w(v) = \frac{3}{4} (1 - v^2) 1(|v| \leq 1)$, and choose the bandwidth by the least-squares cross-validation. The R^2 's of the model have been increased up to 0.66, 0.65, 0.62, 0.68, respectively for the four years.

Figure 2 reports the estimated $a_1(\textit{Experience}, \textit{Region}, :)$ and $a_2(\textit{Experience}, \textit{Region}, :)$ of model (4.2) for different regions averaged across all other categorical variables. $a_1(\textit{Experience}, \textit{Region}, :)$ can be viewed as the direct effects of experience on wage for the particular region (averaged across all other categorical variables), and $a_2(\textit{Experience}, \textit{Region}, :)$ represents the marginal return to schooling as a function of experience for the particular region. We summarize some interesting findings from figure 2. First, while there are considerable variations between regions, we find the direct effects of experience on wage are usually **positive** (upward sloping) but **not necessarily concave**, which is in sharp contrast with the results of the parametric model. Notably, the experience-wage profile estimated here are from cross-sections and cannot be taken as individuals life-cycle earning trend. Second, if the standard Mincer equation holds, we expect the estimated $a_2(\textit{Experience}, \textit{Region}, :)$ to be a horizontal line. But clearly, this is far from reality. The effects of experience on return to schooling are **mainly negative**, which agrees with our previous results from the parametric setting, presented in Table 4. The findings here have interesting econometric interpretation. On the one hand,

we may wonder if higher education causes higher return to seniority, or similarly, longer experience leads to higher return to education. On the other hand, it is possible that the young cohorts (implied by shorter experience) have higher return to education, due to cohort supply effects, technological changes or some other reasons. And we need to resort to empirical results to evaluate the overall influence. In the sample studied here, the later force has been found to dominate the former in their direction of impacts. Admittedly, the interacting patterns of the regressors in the wage equation uncovered by this functional coefficient model require further careful investigation.

Figure 3 reports the estimated $a_1(\textit{Experience}, \textit{Race}, :)$ and $a_2(\textit{Experience}, \textit{Race}, :)$ of model (4.2) for different races averaged across all other categorical variables. $a_1(\textit{Experience}, \textit{Race}, :)$ can be viewed as the direct effects of experience on wage for the race, and $a_2(\textit{Experience}, \textit{Race}, :)$ represents the marginal return to schooling as a function of experience for the particular race. The findings are similar to those in figure 2. We only mention that the return to schooling seems much higher for White and others (above 0.1 across 2/3 of the range of experience) than Hispanic (below 0.1 in almost all the range of experience).

Figure 4 reports the estimated $a_1(\textit{Experience}, :)$ and $a_2(\textit{Experience}, :)$ depending on whether a person is male or female, single or non-single, and veteran or non-veteran. Figure 5 reports the estimated $a_1(\textit{Experience}, \textit{Industry}, :)$ and $a_2(\textit{Experience}, \textit{Industry}, :)$ of model (4.2) for different industries averaged across all other categorical variables. Both figures can be interpreted similarly to the case of figure 2. The most eminent implication by these figures is that return to education does depend heavily upon other variables. In particular, the top panel in figure 4 indicates that higher return to education for female across all the range of age or work experience. In addition, we can see substantial variation among the cells which suggests the highly complex functional form of the wage equation.

Figure 6 reports the estimated $a_1(\textit{Experience}, :)$ and $a_2(\textit{Experience}, :)$ averaged over all categorical variables. Similarly to the cases of figures 2-5, we observe that the direct impact of experience on wage is **positive** but the marginal return to schooling as a function of experience tends to be decreasing except when experience is low (≤ 4 years in 1990, ≤ 12 in 2005). When experience is larger than 37 years, the marginal return to schooling is diminishing very fast a function of experience. Prior to 37 years, the marginal returns to schooling may vary from 0.105 to 0.145.

Therefore, our empirical application has demonstrated the usefulness of our proposed model in uncovering complicated patterns of interacting effects of the covariates on the dependent variable. And the results are of interesting economic interpretation.

5 Conclusions

This paper proposes a local linear functional coefficient estimator that admits a mix of discrete and continuous data for stationary time series. Under weak conditions our estimator is asymptotically normally distributed. We also include simulations and empirical applications. We find from the simulations that our nonparametric estimators behave reasonably well for a variety of DGPs.

As an empirical application, we estimate a human capital earning function from the recent CPS

data. Unlike the widely used linear separable model, or the frequency approach that conducts estimation in splitted samples, the proposed model enables us to utilize the entire dataset and allows the return to education to vary with the other categorical and continuous variables. The empirical findings show considerable interacting effects among the regressors in the wage equation. For instance, the younger cohorts are found to have higher return to education. While these patterns need further explanation from labor economic theory, the application demonstrates the usefulness of our proposed functional coefficient model due to its flexibility and clear economic interpretation. And thus the model has good potential for applied research. Our future research will address some related problems such as the optimal selection of smoothing parameters. Another extensions is to study the estimation of functional coefficient model with both endogeneity and mixed regressors.

6 Appendix: Proof of Theorem 2.1

We use $\|\cdot\|$ to denote the Euclidean norm of \cdot , C to signify a generic constant whose exact value may vary from case to case, and a' to denote the transpose of a . Let $d_{u_i, u} = \sum_{t=1}^q 1(U_{it}^d \neq u_t^d)$, where $1(U_{it}^d \neq u_t^d)$ is an indicator function that takes value 1 if $U_{it}^d \neq u_t^d$ and 0 otherwise. So $d_{u_i, u}$ indicates the number of disagreeing components between U_{it}^d and u_t^d .

We first define some notation. For any $p \times 1$ vectors $c = (c_1, \dots, c_p)'$ and $d = (d_1, \dots, d_p)'$, let $c \odot d = (c_1 d_1, \dots, c_p d_p)'$ and $c/d = (c_1/d_1, \dots, c_p/d_p)'$ whenever applicable. Let

$$S_n = S_n(u) = \begin{pmatrix} S_{n,0} & S_{n,1} \\ S'_{n,1} & S_{n,2} \end{pmatrix}, \quad T_n = T_n(u) = T_{n,1} + T_{n,2},$$

with

$$\begin{aligned} S_{n,0} &= S_{n,0}(u) = n^{-1} \sum_{i=1}^n X_i X_i' K_{iu}, \\ S_{n,1} &= S_{n,1}(u) = n^{-1} \sum_{i=1}^n \left(X_i X_i' \right) \otimes ((U_i^c - u^c)/h)' K_{iu}, \\ S_{n,2} &= S_{n,2}(u) = n^{-1} \sum_{i=1}^n \left(X_i X_i' \right) \otimes (((U_i^c - u^c)/h) ((U_i^c - u^c)/h)') K_{iu}, \\ T_{n,1} &= T_{n,1}(u) = n^{-1} \sum_{i=1}^n \begin{pmatrix} X_i \varepsilon_i \\ (X_i \varepsilon_i) \otimes ((U_i^c - u^c)/h) \end{pmatrix} K_{iu}, \text{ and} \\ T_{n,2} &= T_{n,2}(u) = n^{-1} \sum_{i=1}^n \begin{pmatrix} (X_i X_i' \mathbf{a}(U_i)) \\ (X_i X_i' \mathbf{a}(U_i)) \otimes ((U_i^c - u^c)/h) \end{pmatrix} K_{iu}, \end{aligned}$$

where recall $\mathbf{a}(U_i) = (a_1(U_i), \dots, a_d(U_i))'$. Then

$$\hat{\theta} = H_1^{-1} S_n^{-1} T_n,$$

where $H_1 = \text{diag}(1, \dots, 1, h', \dots, h')$ is a $d(p+1) \times d(p+1)$ diagonal matrix with d diagonal elements

of 1 and d diagonal elements of h . Let $H = \sqrt{nh_1 \dots h_p}$. Then

$$\begin{aligned} HH_1 (\widehat{\theta} - \theta) &= HS_n^{-1} (T_n - S_n \theta) \\ &= HS_n^{-1} T_{n,1} + HS_n^{-1} (T_{n,2} - S_n \theta). \end{aligned}$$

We first prove several lemmas.

Lemma 6.1 (a) $S_{n,0} = \Omega(u) f_u(u) + o_p(1)$,
(b) $S_{n,1} = O_p(\|h\|^2 + \|h\| \|\lambda\|) = o_p(1)$,
(c) $S_{n,2} = \mu_{2,1}(\Omega(u) f_u(u)) \otimes I_p + o_p(1)$.

Proof. We only prove (a) since the proofs of (b) and (c) are similar. First by the stationarity of $\{X_i, U_i\}$,

$$\begin{aligned} E(S_{n,0}) &= E(X_i X_i' K_{iu}) \\ &= E(X_i X_i' W_{h,iu} | d_{u_i u} = 0) p(u^d) + \sum_{s=1}^q E(X_i X_i' W_{h,iu} L_{\lambda,iu} | d_{u_i u} = s) p(d_{u_i u} = s) \\ &= E(\Omega(U_i) W_{h,iu} | d_{u_i u} = 0) p(u^d) + O(\|\lambda\|) \\ &= \int \Omega(u^c + h \odot v, u^d) f_u(u^c + h \odot v, u^d) W(v) dv + O(\|\lambda\|) \\ &= \Omega(u) f_u(u) + O(\|h\|^2 + \|\lambda\|). \end{aligned} \tag{6.1}$$

Since a typical element of $S_{n,0}$ is

$$s_{n,st} = n^{-1} \sum_{i=1}^n X_{is} X_{it} K_{iu}, \quad s, t = 1, \dots, d,$$

by the Chebyshev's inequality, it suffices to show that

$$\text{var}(s_{n,st}) = o(1). \tag{6.2}$$

Let $\xi_i = X_{is} X_{it} K_{iu}$. By the stationarity of $\{X_i, U_i\}$, we have

$$\text{var}(s_{n,st}) = \frac{1}{n} \text{var}(\xi_1) + \frac{2}{n} \sum_{j=1}^{n-1} \left(1 - \frac{j}{n}\right) \text{cov}(\xi_1, \xi_j). \tag{6.3}$$

Clearly,

$$\text{var}(\xi_1) \leq E(X_{1s}^2 X_{1t}^2 K_{1,u}^2) = O((h_1 \dots h_n)^{-1}). \tag{6.4}$$

To obtain an upper bound for the second term on the right hand side of (6.3), we split it into two terms as follows

$$\sum_{j=1}^{n-1} |\text{cov}(\xi_1, \xi_j)| = \sum_{j=1}^{d_n} |\text{cov}(\xi_1, \xi_j)| + \sum_{j=d_n+1}^{n-1} |\text{cov}(\xi_1, \xi_j)| \equiv J_1 + J_2,$$

where d_n is a sequence of positive integers such that $d_n h_1 \dots h_p \rightarrow 0$ as $n \rightarrow \infty$. Since for any $j > 1$,

$$|E(\xi_1 \xi_j)| = |E(X_{1s} X_{1t} K_{1,u} X_{js} X_{jt} K_{j,u})| = O(1),$$

$J_1 = O(d_n)$. For J_2 , by the Davydov's inequality (e.g., Hall and Heyde, 1980, p. 278; or Bosq, 1996, p. 19), we have

$$\begin{aligned} \text{cov}(\xi_1, \xi_j) &\leq C [\alpha(j-1)]^{\gamma/(2+\gamma)} \left(E |\xi_1|^{2+\gamma} \right)^{2/(2+\gamma)} \\ &= C [\alpha(j-1)]^{\gamma/(2+\gamma)} \left\{ E \left| (X_{1s} X_{1t})^{(2+\gamma)} K_{1,u}^{2+\gamma} \right| \right\}^{2/(2+\gamma)} \\ &= O \left((h_1 \dots h_p)^{-(2+2\gamma)/(2+\gamma)} \right) [\alpha(j-1)]^{\gamma/(2+\gamma)}. \end{aligned} \quad (6.5)$$

So

$$\begin{aligned} J_2 &\leq C (h_1 \dots h_p)^{-(2+2\gamma)/(2+\gamma)} \sum_{j=d_n}^{n-1} [\alpha(j)]^{\gamma/(2+\gamma)} \\ &\leq C (h_1 \dots h_p)^{-(2+2\gamma)/(2+\gamma)} d_n^{-\alpha} \sum_{j=d_n}^{\infty} j^\alpha [\alpha(j)]^{\gamma/(2+\gamma)} = o \left((h_1 \dots h_p)^{-1} \right), \end{aligned} \quad (6.6)$$

by choosing d_n such that $d_n^{-\alpha} (h_1 \dots h_p)^{-\gamma/(2+\gamma)} = o(1)$. This, in conjunction with (6.3)-(6.4), implies, $\text{var}(s_{n,st}) = O \left((nh_1 \dots h_p)^{-1} \right) = o(1)$. ■

Lemma 6.2

$$HT_{n,1} = n^{-1/2} (h_1 \dots h_p)^{1/2} \sum_{i=1}^n \begin{pmatrix} X_i \varepsilon_i \\ (X_i \varepsilon_i) \otimes ((U_i^c - u^c)/h) \end{pmatrix} K_{iu} \xrightarrow{d} N(0, \Gamma),$$

where $H = \sqrt{nh_1 \dots h_p}$, $\sigma^2(u, x) = E[\varepsilon_i^2 | U_i = u, X_i = x]$, $\Omega^*(u) = E[X_i X_i' \sigma^2(U_i, X_i) | U_i = u]$, and

$$\Gamma = \Gamma(u) = f_u(u) \begin{pmatrix} \mu_{0,2}^p \Omega^*(u) & 0' \\ 0 & \mu_{2,2} \Omega^*(u) \otimes I_p \end{pmatrix}.$$

Proof. Let c be a unit vector on $\mathbb{R}^{d(p+1)}$. Let

$$\zeta_i = (h_1 \dots h_p)^{1/2} c' \begin{pmatrix} X_i \varepsilon_i \\ (X_i \varepsilon_i) \otimes ((U_i^c - u^c)/h) \end{pmatrix} K_{iu}.$$

By the Cramér-Wold device, it suffices to prove

$$I_n = n^{-1/2} \sum_{i=1}^n \zeta_i \xrightarrow{d} N(0, c' \Gamma c). \quad (6.7)$$

Clearly, by the law of iterated expectation, $E(\zeta_i) = 0$. Now

$$\text{var}(I_n) = \text{var}(\zeta_1) + 2 \sum_{j=1}^{n-1} \left(1 - \frac{j}{n}\right) \text{cov}(\zeta_1, \zeta_j).$$

By arguments similar to those used in the proof of Lemma 6.1,

$$\begin{aligned} & \text{var}(\zeta_1) \\ = & h_1 \dots h_p c' E \left\{ \left(\begin{array}{cc} \Omega^*(U_i) & \Omega^*(U_i) \otimes ((U_i^c - u^c)/h)' \\ \Omega^*(U_i) \otimes ((U_i^c - u^c)/h) & \Omega^*(U_i) \otimes (((U_i^c - u^c)/h)((U_i^c - u^c)/h)') \end{array} \right) K_{iu} \right\} c \\ = & c' \Gamma c + o(1), \end{aligned}$$

and

$$\sum_{j=1}^{n-1} |\text{cov}(\zeta_1, \zeta_j)| = o(1),$$

which implies that

$$\text{var}(I_n) \rightarrow c' \Gamma c \text{ as } n \rightarrow \infty.$$

Using the standard Doob's small-block and large-block technique, we can finish the rest of the proof by following the arguments of Cai, Fan and Yao (2000, pp.954-955) or Cai and Ould-Saïd (2003, pp.446-448). ■

Lemma 6.3 *Let $B_n = H(T_{n,2} - S_n \theta)$. Then $B_n = b(h, \lambda) + o_p(1)$, where $b(h, \lambda)$ is defined in (2.9).*

Proof. Let

$$\begin{aligned} \varsigma_i = & H \left(\begin{array}{c} (X_i X_i' \mathbf{a}(U_i)) \\ (X_i X_i' \mathbf{a}(U_i)) \otimes ((U_i^c - u^c)/h) \end{array} \right) K_{iu} \\ & - H \left(\begin{array}{cc} (X_i X_i') & (X_i X_i') \otimes ((U_i^c - u^c)/h)' \\ (X_i X_i') \otimes ((U_i^c - u^c)/h) & (X_i X_i') \otimes (((U_i^c - u^c)/h)((U_i^c - u^c)/h)') \end{array} \right) \theta K_{iu}. \end{aligned}$$

Then we have

$$B_n = \frac{1}{n} \sum_{i=1}^n \varsigma_i. \quad (6.8)$$

Let $\bar{\varsigma}_i = E(\varsigma_i | U_i)$. Then

$$\begin{aligned} E(B_n) &= E(\bar{\varsigma}_i) \\ &= E\{\bar{\varsigma}_i | d_{u_i u} = 0\} p(u^d) + E\{\bar{\varsigma}_i | d_{u_i u} = 1\} P(d_{u_i u} = 1) + O(H \|\gamma\|^2) \\ &\equiv b_{n,1} + b_{n,2} + o(1). \end{aligned}$$

On the set $\{U_i^d = u^d, W_{h,iu} > 0\}$,

$$a_j(U_i) = a_j(u) + \dot{a}_j(u)' (U_i^c - u^c) + \frac{1}{2} (U_i^c - u^c)' \ddot{a}_j(u) (U_i^c - u^c) + o(\|h\|^2).$$

Let $A(U_i, u) = ((U_i^c - u^c)' \ddot{a}_1(u) (U_i^c - u^c), \dots, (U_i^c - u^c)' \ddot{a}_d(u) (U_i^c - u^c))'$. Recall $A = (\sum_{s=1}^p h_s^2 a_{1,ss}(u), \dots, \sum_{s=1}^p h_s^2 a_{d,ss}(u))'$, and $\mathbf{b}(u) = (\dot{a}_1(u)', \dots, \dot{a}_d(u)')$. Then we have

$$\begin{aligned} b_{n,1} &= \frac{1}{2} H E \left\{ \left(\begin{array}{c} \Omega(U_i) A(U_i, u) \\ (\Omega(U_i) A(U_i, u)) \otimes ((U_i^c - u^c)/h) \end{array} \right) W_{h,iu} \Big| d_{u_i u} = 0 \right\} \times p(u^d) + o(1) \\ &= \frac{H \mu_{2,1}}{2} \begin{pmatrix} f_u(u) \Omega(u) A \\ 0 \end{pmatrix} + o(1), \end{aligned}$$

and

$$\begin{aligned}
& b_{n,2} \\
&= H E \{ \bar{\zeta}_i | d_{u_i u} = 1 \} P(d_{u_i u} = 1) \\
&= H E \left\{ \left(\begin{array}{l} \Omega(U_i) (\mathbf{a}(U_i) - \mathbf{a}(u)) - (\Omega(U_i) \otimes ((U_i^c - u^c)/h)') \mathbf{b}(u) \\ (\Omega(U_i) (\mathbf{a}(U_i) - \mathbf{a}(u))) \otimes ((U_i^c - u^c)/h) - (\Omega(U_i) \otimes ((U_i^c - u^c)/h) ((U_i^c - u^c)/h)') \mathbf{b}(u) \end{array} \right) \right. \\
&\quad \left. \times K_{iu} \Big| d_{u_i u} = 1 \right\} p(d_{u_i u} = 1) + o(1) \\
&= H \sum_{s=1}^q \lambda_s I_s(u^d, \tilde{u}^d) f_u(u^c, \tilde{u}^d) \left(\begin{array}{l} \Omega(u^c, \tilde{u}^d) (\mathbf{a}(u^c, \tilde{u}^d) - \mathbf{a}(u)) \\ -\mu_{2,1} (\Omega(u^c, \tilde{u}^d) \otimes I_p) \mathbf{b}(u) \end{array} \right) + o(1).
\end{aligned}$$

Consequently, $E(B_n) = b(h, \lambda) + o(1)$, where $b(h, \lambda)$ is defined in (2.9).

To show $\text{var}(B_n) = o(1)$ elementwise, we focus on the first d elements $\zeta_i^{(1)}$ of ζ_i since the other cases are similar, where

$$\zeta_i^{(1)} = H [X_i X_i' (\mathbf{a}(U_i) - \mathbf{a}(u)) - (X_i X_i' \otimes ((U_i^c - u^c)/h)') \mathbf{b}(u)] K_{iu}.$$

A typical element of $\zeta_i^{(1)}$ is

$$\zeta_{i,t}^{(1)} = H \left[X_{it} \sum_{s=1}^d X_{is} (a_s(U_i) - a_s(u)) - X_{it} \sum_{s=1}^d X_{is} ((U_i^c - u^c)/h)' b_j(u) \right] K_{iu},$$

$t = 1, \dots, d$.

$$\text{var} \left(\frac{1}{n} \sum_{i=1}^n \zeta_{i,t}^{(1)} \right) = \frac{1}{n} \text{var} \left(\zeta_{1,t}^{(1)} \right) + \frac{2}{n} \sum_{j=1}^{n-1} \left(1 - \frac{j}{n} \right) \text{cov} \left(\zeta_{1,t}^{(1)}, \zeta_{j,t}^{(1)} \right).$$

By arguments similar to those used in the proof of Lemma 6.1,

$$\frac{1}{n} \text{var} \left(\zeta_{1,t}^{(1)} \right) = O \left(\|h\|^4 + \|\lambda\|^2 \right) = o(1),$$

and

$$\sum_{j=1}^{n-1} \left| \text{cov} \left(\zeta_{1,t}^{(1)}, \zeta_{j,t}^{(1)} \right) \right| = o(1),$$

which implies that $\text{var} \left(\frac{1}{n} \sum_{i=1}^n \zeta_{i,t}^{(1)} \right) = o(1)$. Similarly, one can show that the variance of the other elements in B_n is $o(1)$. The conclusion then follows by the Chebyshev's inequality. ■

By Lemmas 6.1-6.3,

$$HH_1 \left(\hat{\theta} - \theta \right) - B^{-1} b(h, \lambda) \xrightarrow{d} N \left(0, B^{-1} \Gamma B^{-1} \right).$$

This completes the proof.

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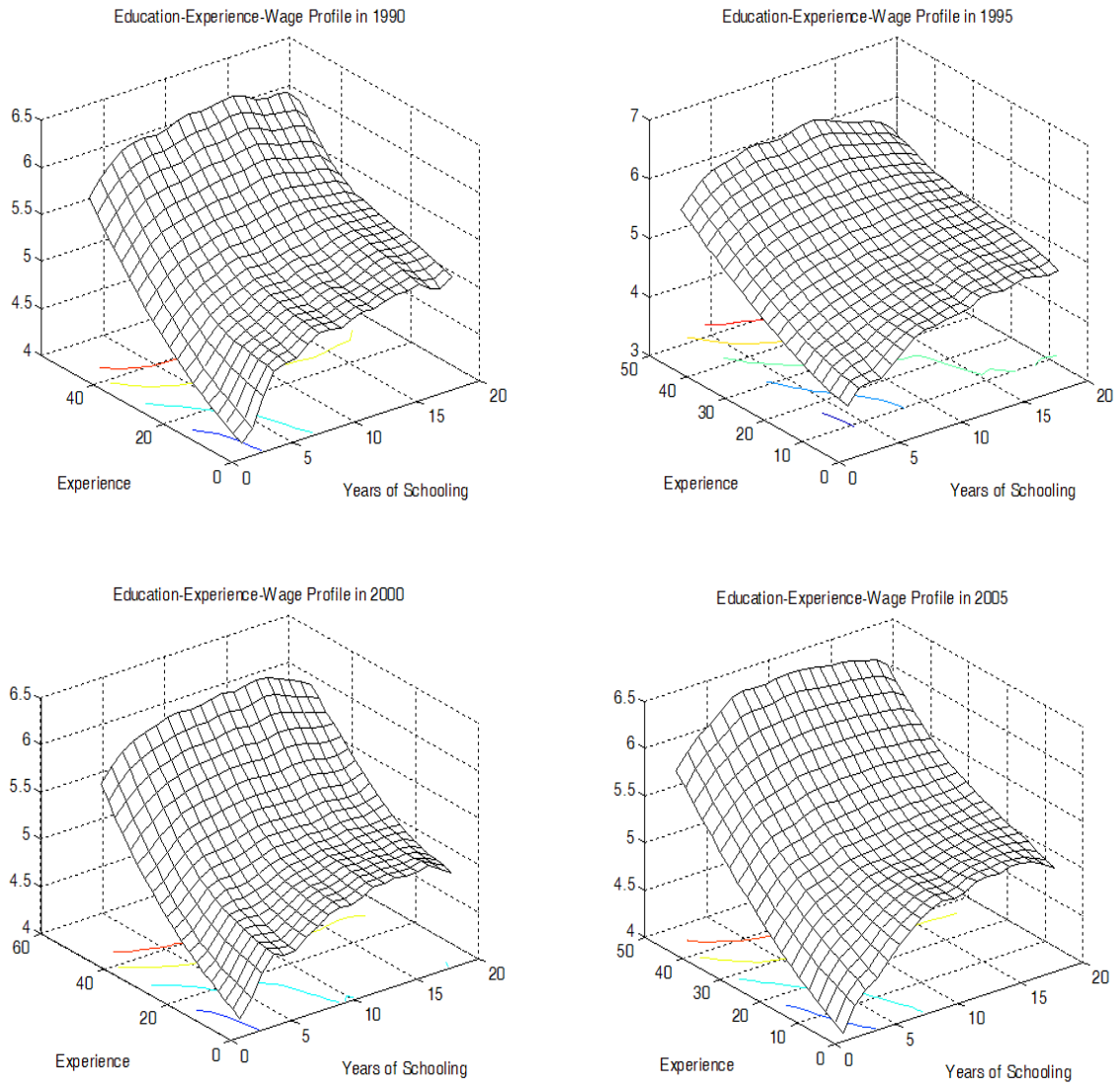


Figure 1: Education-Experience-Wage profile resulting from the partially linear models

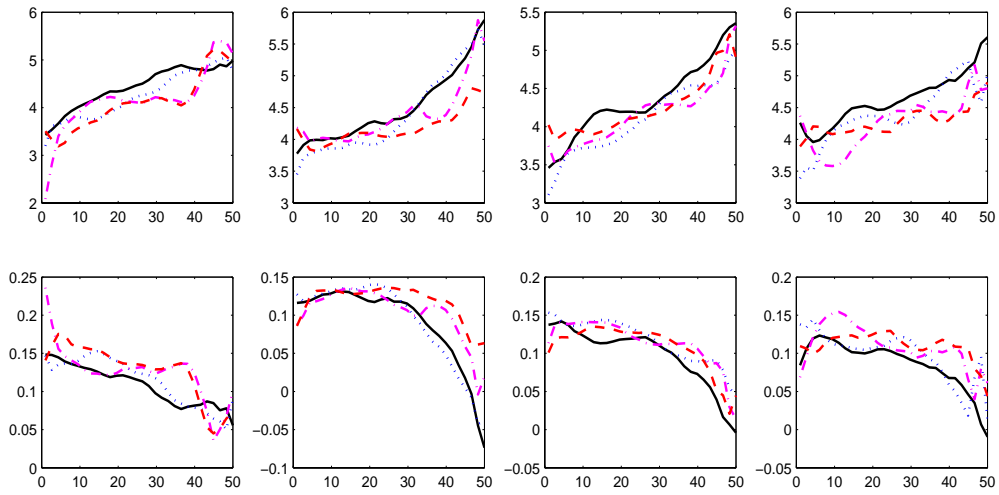


Figure 2: Plots of $a_1(Experience, Region, :)$ and $a_2(Experience, Region, :)$ averaging over other categorical variables. Horizontal axis: *Experience*. Vertical axis: a_1 or a_2 . The two rows correspond to a_1 and a_2 respectively from the top to the bottom. The four columns correspond to *Region* = Northeast, Midwest, South and West from the left to the right column. 1990: solid line, 1995: dotted line, 2000: dashdot line, 2005: dashed line.

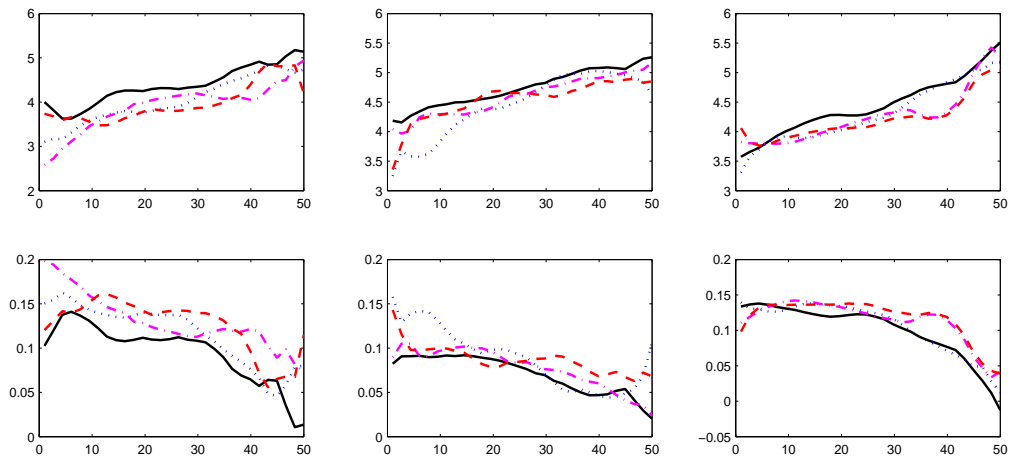


Figure 3: Plots of $a_1(\textit{Experience}, \textit{Race}, :)$ and $a_2(\textit{Experience}, \textit{Race}, :)$ averaging over other categorical variables. Horizontal axis: *Experience*. Vertical axis: a_1 or a_2 . The two rows correspond to a_1 and a_2 from the top to the bottom. The three columns correspond to *Race* = Otherwise, Hispanic, and White from the left to the right column. 1990: solid line, 1995: dotted line, 2000: dashdot line, 2005: dashed line.

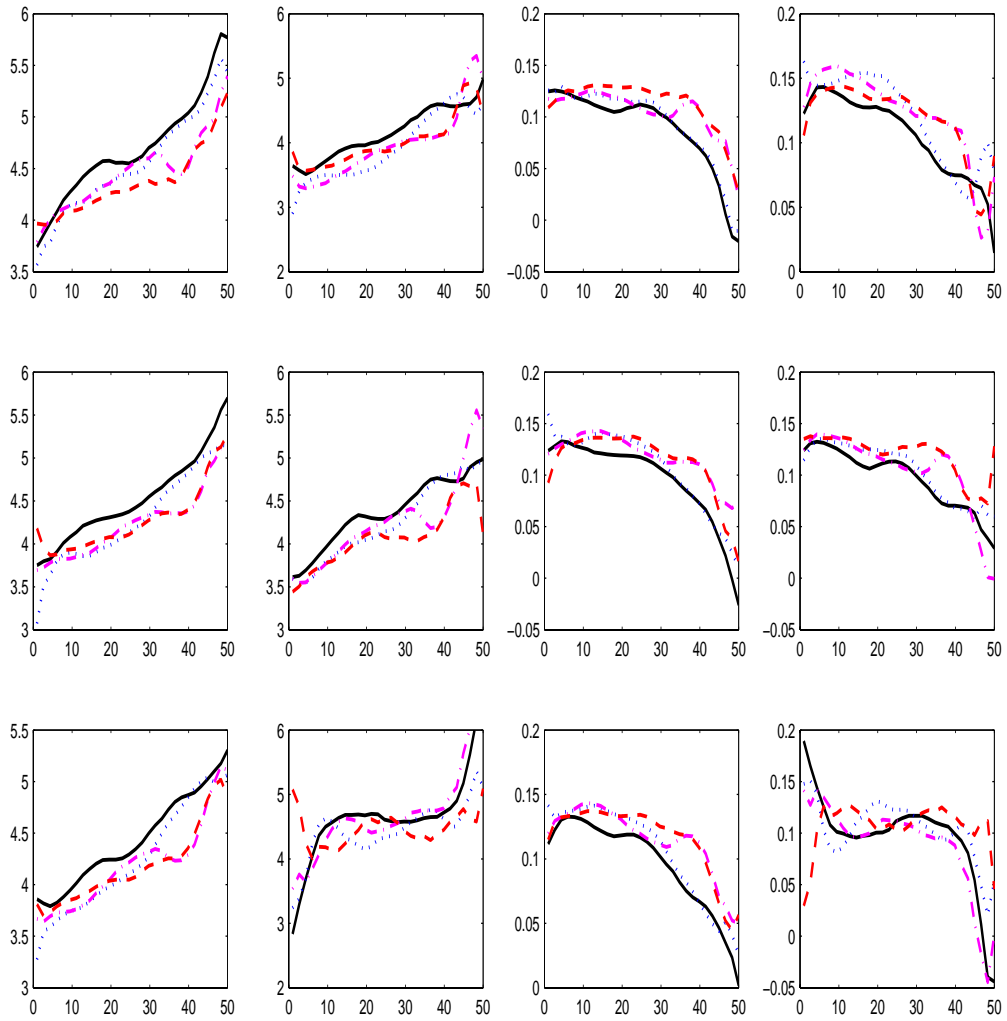


Figure 4: Plots of $a_1(\text{Experience}, \text{Gender}, :)$ and $a_2(\text{Experience}, \text{Gender}, :)$ (1st row), $a_1(\text{Experience}, \text{Single}, :)$ and $a_2(\text{Experience}, \text{Single}, :)$ (2nd row), $a_1(\text{Experience}, \text{Veteran}, :)$ and $a_2(\text{Experience}, \text{Veteran}, :)$ (3rd row), averaging over other categorical variables. Horizontal axis: *Experience*. Vertical axis: a_1 or a_2 . First row: the four columns from the left to the right correspond to a_1 for male, a_1 for female, a_2 for male, and a_2 for female, respectively. Second row: the four columns from the left to the right correspond to a_1 for non-single, a_1 for single, a_2 for non-single, and a_2 for single, respectively. Third row: the four columns from the left to the right correspond to a_1 for non-veteran, a_1 for veteran, a_2 for non-veteran, and a_2 for veteran, respectively. 1990: solid line, 1995: dotted line, 2000: dashdot line, 2005: dashed line.

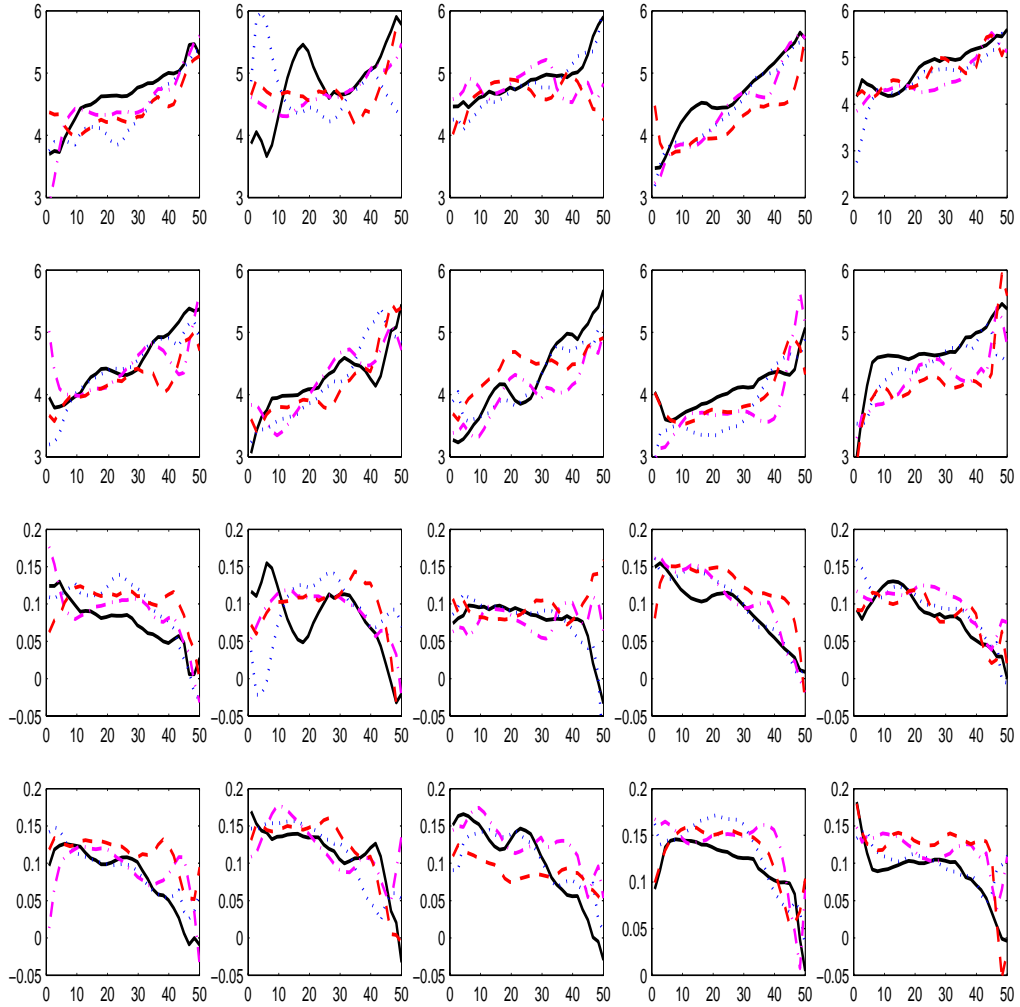


Figure 5: Plots of $a_1(\text{Experience}, \text{Industry}, :)$ and $a_2(\text{Experience}, \text{Industry}, :)$ averaging over other categorical variables. Horizontal axis: *Experience*. Vertical axis: a_1 or a_2 . The first two rows correspond to a_1 , and the last two rows correspond to a_2 . For rows 1 and 3, the five columns from the left to the right correspond respectively to *Industry* = Agriculture, Mining, Construction, Manufacturing, and Transportation. For rows 2 and 4, the five columns from the left to the right correspond respectively to *Industry* = Wholesale and return, Finance, Personal services, Professional services, and Public administration. 1990: solid line, 1995: dotted line, 2000: dashdot line, 2005: dashed line.

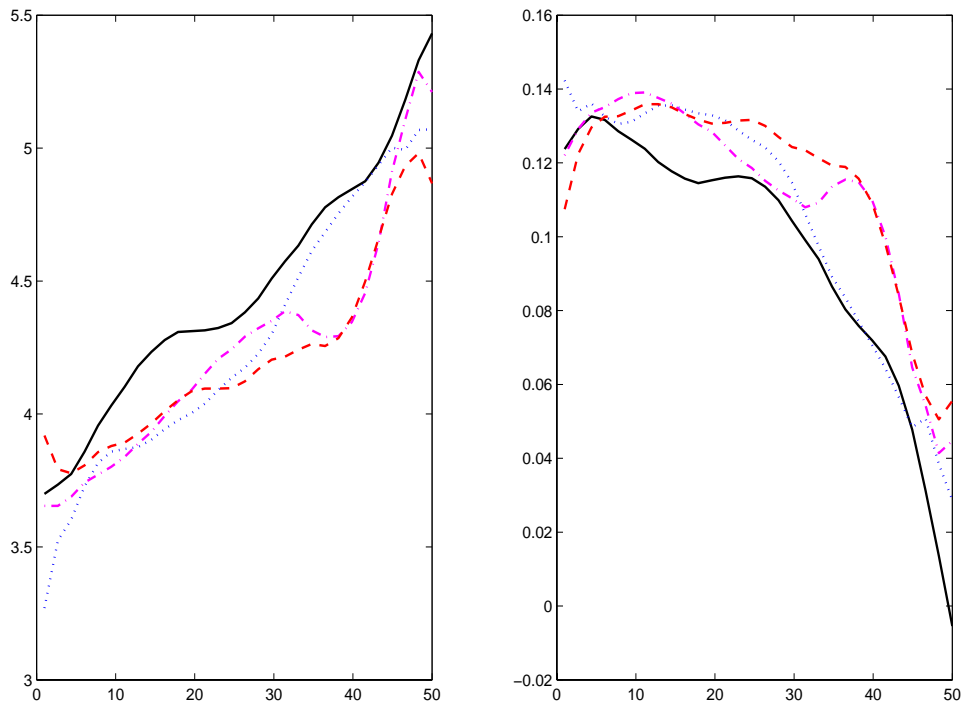


Figure 6: Plots of a_1 ($Experience, :$) and a_2 ($Experience, :$) averaging over all categorical variables. Horizontal axis: $Experience$. Vertical axis: a_1 or a_2 . The two columns from the left to the right correspond to a_1 and a_2 , respectively. 1990: solid line, 1995: dotted line, 2000: dashdot line, 2005: dashed line.