An exact analysis of the consumption CAPM

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Abstract

In this paper we take up Bayesian inference for the consumption capital asset pricing model. The model has several econometric complications. First, it implies exact relationships between asset returns and the endowment growth rate that will be rejected by all possible realizations. Second, it was thought before that it is not possible to express asset returns in closed form. We show that Labadie's (1989) solution procedure can be applied to obtain asset returns in closed form and, therefore, give an econometric interpretation in terms of traditional measurement error models. We apply the Bayesian inference procedures to the Mehra and Prescott (1985) data set, we provide posterior distributions of structural parameters and posterior predictive asset return distributions, and we use these distributions to assess the existence of asset returns puzzles.

JEL codes: C11, C23.

Key Words: Asset pricing, Bayesian inference, Markov Chain Monte Carlo, equity premium puzzle.

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1. Introduction

The consumption capital asset pricing model (C-CAPM) has played a central role in the modern theory of finance and macroeconomics. It has been used by Mehra and Prescott (1985) to investigate the so-called "equity premium puzzle": Over the past hundred years, the average real stock return in the United States has been about seven per cent and the average real return on Treasury bills has been about one percent. In their seminal paper, Mehra and Prescott (1985) showed that the C-CAPM is unable to replicate these facts which we refer to as "asset return puzzles". Kocherlakota's (1996) in a thorough literature review concludes that only market frictions, incompleteness, or non-standard preferences could solve the equity premium and risk-free rate puzzles. Moreover, there is an abundance of findings in the GMM literature that the C-CAPM is rejected under fully non-parametric conditions. Indeed, Hansen, Heaton, and Yaron (1996) reject even after elaborate corrections for the finite-sample properties of GMM tests. It is also known, however, that the finite sample properties of GMM could be very different compared to their asymptotic properties.

For reasons lucidly explained in the excellent paper by Geweke (1999) an exact likelihood-based analysis of the C-CAPM, is very complicated. The model implies exact relationships between the endogenous variables as there are many endogenous variables but only a small number of shocks. Therefore, the model will be rejected by every possible data set. In that sense, the GMM-based frequent rejections of the C-CAPM are not surprising at all. Several attempts have been made to provide parameter estimates for the model, including calibration, variants of indirect inference, GMM methods etc. All of them solve part of the problem since they do not confront the data with a model-based exact likelihood function to provide inferences.

The fundamental reason behind the widespread application of GMM methods in asset pricing models is the presumption that the implications of C-CAPM are exhausted in the provision of first-order Euler conditions that cannot be solved to obtain closed-form expressions for asset returns. In fact, this is not true, since Labadie (1989) has shown that Euler equations can be solved to provide closed-form expressions. Since closed-form expressions are available, inferences based on the likelihood function, appropriately constructed, should be possible. The purpose of this
paper is to explore the implications of this idea, and apply the results to the Mehra and Prescott (1985) data set to assess the existence of an "equity premium puzzle" using formal means\(^1\). Artificial data sets are also considered to validate the new methods presented here, and examine their sensitivity.

2. The asset pricing model

Consider an economy with a representative agent endowed with constant relative risk aversion preferences. The preferences of the representative agent are \( E_0 \sum_{t=0}^{\infty} \beta^t u(c_t) \) where \( \beta > 0 \) is the discount factor, and the spot utility function is given by:

\[
u(c) = \begin{cases} c^{1-\gamma} - \frac{1}{1-\gamma}, & \gamma > 0, \gamma \neq 1 \\ \log(c), & \gamma \neq 1, \end{cases}
\]

(1)

where \( \gamma \) is the coefficient of relative risk aversion (RRA), and \( c_t \) is consumption of date \( t \). There is a single productive unit that produces without cost the consumption good each period. Let \( Y_t \) be the date \( t \) endowment and

\[ y_{t+1} = \frac{Y_{t+1}}{Y_t}. \]

(2)

The stochastic process driving endowment is given by

\[
\log y_{t+1} = \kappa + \delta \log y_t + u_{t+1}, \quad t=0,1,2,\ldots,
\]

(3)

where \( u_{t+1} \sim IN(0, \sigma_u^2) \) is an error term, for \( t=1,2,\ldots \), and \( \kappa, \delta \) are parameters. The reason for indexing the shock with a subscript "1" will become clear later on. The AR(1) specification is supported by evidence in Labadie (1989) and Cecchetti et al. (1990). Let \( q_t \) denote the price of equity and \( P_t \) the discount price of risk-free real bonds in terms of the consumption good. The risk-free real interest rate is denoted by \( R_t^f \) and is given in closed form as:

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\(^1\) Mehra and Prescott (1985) advocate the use of calibration techniques to assess the model in terms of behavior of asset returns. Calibration is widespread practice in the real-business-cycle and asset pricing literature. We do not wish to take a view on whether calibration is or is not useful. Our purpose is solely to explore the implications and capabilities of formal, exact, likelihood-based analysis of the asset pricing model.
The real return on equity is denoted by $R^e_t$, and is given by

$$1 + R^e_t = q_{t+1} + Y_{t+1}, \quad t = 0, 1, 2, \ldots$$

(5)

It can be shown that existence of an equilibrium with finite stock prices, requires certain conditions, for details see Tsionas (2003). Asset returns, can be obtained using a modification of Labadie’s (1989) procedure to obtain them in a series expansion. If $\rho = 1 - \gamma$, define the recursive coefficients

$$A_{j+1} = A_j \beta \exp \left[ \left( a_{j+1} + \rho \right) \left( \kappa + \frac{1}{2} \left( a_{j+1} + \rho \right) \sigma_u^2 \right) \right],$$

$$a_{j+1} = \delta \left( a_j + \rho \right), \quad j = 2, 3, \ldots,$$

(6)

with $A_1 = \beta \exp \left[ \rho \left( \kappa + \frac{1}{2} \rho \sigma_u^2 \right) \right]$, and $a_1 = \rho \delta$.

Then, stock prices are given by:

$$q_t = Y_t \sum_{j=1}^\infty A_j y_t^{a_j}.$$  

(7)

Gross real stock returns are given by the expression:

$$1 + R^q_{t+1} = \frac{1 + \sum_{j=1}^\infty A_j y_{t+1}^{a_j}}{\sum_{j=1}^\infty A_j y_{t+1}^{a_j}}, \quad t = 0, 1, 2, \ldots$$

(8)

and the gross real risk-free rate is provided in equation (4).

3. The econometric model

It is clear that the model has implications for asset returns that should be confronted with the data, not only to assess the model’s ability to replicate the behavior of observed returns but also in order to provide parameter estimates, and allow formal statistical inferences for parameters and functions of interest. Since we are able to express asset returns in closed form, via equations (4) and (8), it is obvious that the likelihood function should be derived based on the system consisting of equations (3), (4) and (8). The problem is, of course, that there is only one shock, namely $u_t$, so the model is not complete, and the singularity problem mentioned in Geweke (1999) exists, thus prohibiting the formulation of a likelihood function and the
conducted of formal statistical inferences. We propose to solve this problem by following
a familiar path in econometrics, namely the view that the endogenous variables are
measured with error, leading us to introduce error terms in equations (4) and (8). However,
first we have to truncate the infinite summations. We do this by computing
the index \( j \) so that \( A_j y^a_j \leq \varepsilon \), where \( \varepsilon \) is a small positive constant. Define \( M_t(\lambda) = j \),
where \( \lambda = [\beta, \gamma, \kappa, \delta, \sigma_u^2]^\prime \) is the parameter vector. Therefore, we obtain

\[
1 + R^a_{t+1} = y_{t+1} + u_{2,t+1} + \sum_{j=1}^{M_t(\lambda)} A_j y^a_j
\]

and the gross real risk-free rate is:

\[
1 + R^f_t = \beta \exp(\gamma \kappa - \frac{1}{2} \gamma^2 \sigma_u^2) y^a_{t+1}^0 + u_{3t}.
\]

We let \( u_t = [u_{1t}, u_{2t}, u_{3t}] \sim \text{IN}(0, \Sigma) \) where \( \Sigma \) is a 3x3 covariance matrix. Denote the
parameter vector by \( \theta = [\beta, \gamma, \kappa, \delta, \text{vec}(\Sigma)] \). The peculiarity of this system is that it is highly nonlinear with respect to \( \theta \), the
structural equations depend explicitly on \( \sigma_{11} = \sigma_u^2 \), and also the number of terms in
each summation, \( M_t(\lambda) \), is parameter-dependent.

We wish to emphasize that our methodological contribution is the recognition that
there are closed form expressions for asset returns given by Labadie's methodology, a
point also emphasized in a somewhat different context by Tsionas (2003, 2005). Previous research in asset pricing models neglected to recognize this fact and resorted
to GMM-based procedures. Since, however, we have closed-form expressions for asset
returns we can proceed to derive a closed-form likelihood function by introducing
measurement error (in the familiar econometric way) in the closed-form expressions
defining asset returns. This is important because it will allow us to propose likelihood-based methods not only for parameter inferences (and, indeed, important parameters
like the coefficient of relative risk aversion) but also for a likelihood-based examination of the celebrated equity premium puzzle.
Suppose we have a sample $X = [X_1', X_2', \ldots, X_T']$, where $X_t = [\ln y_t, R_t', R_t''']$, $t = 1, \ldots, T$. Then, the log likelihood function of the model is

$$L(\theta; X) = -\frac{N}{2} \log 2\pi - \frac{T}{2} \log |\Sigma| - \frac{1}{2} tr[(X_t - X_t(\theta))(X_t - X_t(\theta))\Sigma^{-1}],$$

(11)

where $X_t(\theta)$ represents the theoretical predictions of the model derived from (9) and (10), $\theta = [\beta, \gamma, \kappa, \delta, \text{vech}(\Sigma)]'$ is the parameter vector. The asset pricing model raises several issues in connection with the conduct of formal statistical inference. First, how should we formulate the priors? Second, how sensitive are the posterior results to different priors? Third, how should we evaluate the model’s ability to reproduce properties of observed asset returns, and assess the existence of an “equity premium puzzle”? Fourth, what can we say about the performance of the Bayesian methods proposed in this paper when applied to artificial samples? These are the questions that we set out to answer in the following discussion.

By application of Bayes’ theorem, we obtain the posterior distribution, $p(\theta | X) \propto L(\theta; X)p(\theta)$, where $p(\theta)$ represents the prior. Our objective is to make inferences on parameters and functions of interest. Important functions of interest in our case, are asset returns that can be used to assess formally the asset return puzzles. We use Markov Chain Monte Carlo techniques to generate a sample $\{\theta^{(s)}, s = 1, \ldots, R\}$ that converges to the distribution whose density is $p(\theta | X)$. We use an adaptive Metropolis algorithm that updates each component of $\theta$ sequentially (so that it resembles a Gibbs sampling with complete blocking). We have $p = 10$ parameters in total. For each $i = 1, \ldots, p$, we generate a candidate $\theta_i^*$ from a proposal distribution that is uniform in the interval $[a_i, b_i]$. With probability $A(\theta^{(r)}, \theta^*) = \min\left(1, \frac{p(\theta^* | X)}{p(\theta^{(r)} | X)}\right)$ we set $\theta_i^{(r+1)} = \theta_i^*$, else we set $\theta_i^{(r+1)} = \theta_i^{(r)}$. We determine the $a_i$ and $b_i$ during a preliminary run to ensure that the acceptance rate is not too high or too low. In the application reported in section 5 we set the desired acceptance rate to 50%. The final acceptance rate in the after-burn-in sample was close to 30%. The candidate parameter vector,
\( \theta' \), has to satisfy several constraints. First, the resulting matrix \( \Sigma \) must be positive definite. This can be ensured by reparameterizing using \( \Sigma = C'C \), where \( C \) is the Cholesky decomposition, and its elements are unrestricted. Second, we must have 
\[ |\delta| < 1. \]
Third, the resulting \( \theta \) must be consistent with existence of an equilibrium with finite stock prices. Because of (6) and (7) the relevant conditions are
\[ \beta \exp[\zeta(\kappa + \frac{1}{2}\zeta\sigma^2)] < 1, \text{ and } |\zeta| < 1, \]
where \( \zeta = \rho/(1-\delta) \). Fourth, we have the constraints
\[ 0 < \beta < 1, \text{ and } \gamma > 0. \]
We use the obvious rejection technique to accommodate these constraints. Denote the admissible parameter space by \( \Theta \subset R^\rho \).

4. Prior elicitation

It is known that the Jeffreys' prior is a form of “non-informative” prior in the context of many econometric models. We have evaluated the Jeffreys' prior by numerical means and we report the results in Appendix A of this paper. We conclude that the Jeffreys' prior is not a reasonable prior for the coefficient of relative risk aversion and the discount factor in the context of asset pricing or real business cycle models. For this reason, we have to adopt a different, reasonable prior for the parameters.

The parameters of interest are \( \beta, \gamma, \kappa, \delta, vech(\Sigma)' \). The discount factor, \( \beta \), should be in the interval \( (0, 1) \) and values in excess of about 0.8 are highly likely. Therefore, as its prior distribution, we assume a Beta (20,1). According to the real business cycle studies, the constant relative risk aversion parameter, \( \gamma \), should be between 1 and 10, and values between 1 and 3 seem to be quite successful in reproducing the business cycle facts so such values should be considered more plausible. We choose a lognormal prior, \( \ln \gamma \sim N(\mu_0, \sigma_\gamma^2) \).
Setting $\sigma_0^2=1$, we examine the prior distribution of $\gamma$, for two different values of $\mu_0$, -1 and 1.445, which give $E(\gamma)=1$ and $E(\gamma)=7$, respectively. For the parameters $\kappa$, and $\delta$, we use independent normal priors. We assume that $\kappa \sim N(0,0.025)$ and $\delta \sim N(0,0.0025)$. These priors, reflect our prior notion that these parameters are likely to be small, that for $\kappa$ we do not wish to be particularly informative, and that with probability 95%, parameter $\delta$ lies in the interval from -0.05 to 0.05. These values, are quite reasonable in view of the fact that these parameters refer to an AR(1) process for output growth. For $\sigma_u$, we have an inverted gamma distribution, $\sigma_u \sim IG(\alpha,\beta)$ with density

$$p(\sigma_u) = \frac{b^a}{\Gamma(a)} \sigma_u^{-(a+1)} e^{-b/\sigma_u}, \quad a, b, \sigma_u > 0.$$ 

We examine two cases, $(a,b)=(2.11,0.011)$ and $(a,b)=(4.77,0.19)$. In this way, the variance of $\sigma_u$ is 0.03$^2$ whereas the mean is 0.01 and 0.05, respectively. For the remaining elements of $\Sigma$, we are completely uninformative, therefore we have flat priors. The priors are presented graphically in Figures 1 through 3.

![Figure 1: Prior distributions of discount factor, $\beta$.](image-url)
Figure 2: Prior distributions of relative risk aversion, $\gamma$. 
Although we believe the priors are reasonable, it is important to examine their implications in some detail. The natural way to examine the implications is to investigate what the priors imply about asset returns. So far, we have assumed a prior distribution \( p(\theta | \xi) \) for the parameters of interest, \( \theta \), conditional on certain values for the hyper-parameters, \( \xi \). It is possible to draw a random sample of values, say \( \{\theta^{(m)}, m = 1, \ldots, M\} \) from the prior \( p(\theta | \xi) \), for some fixed choice of \( \xi \). For each \( \theta^{(m)} \), we can use equations (4) and (5) to draw samples \( \{R_t^{f,(m)}, R_t^{q,(m)}, t = 1, \ldots, T\} \) corresponding to each parameter draw. Denote \( \overline{R}_t^{f,(m)} = T^{-1} \sum_{t=1}^{T} R_t^{f,(m)} \) and \( \overline{R}_t^{q,(m)} = T^{-1} \sum_{t=1}^{T} R_t^{q,(m)} \), the sample means of risk-free rate and equity return for each parameter draw. The joint distribution of \( \overline{R}_t^{q} \) and \( \overline{R}_t^{f} \) can be estimated using the draws \( \overline{R}_t^{f,(m)} \) and \( \overline{R}_t^{q,(m)} \), \( m = 1, \ldots, M \). Clearly, this is an approximation to the prior predictive distribution of
asset returns, and the approximation becomes better as the number of simulations, $M$, increases.

Using our distributional assumptions on priors, and fixing the discount factor to 0.98, we draw 5,000 random samples from the priors, to use them as parameters. We can use (4) and (5) to evaluate the returns. From each evaluation, we save the mean of the equity and bond returns. The joint distributions of average returns are presented in Figure 4. The message from these implied priors, is that they are concentrated around large values of the equity premium and large values of the risk-free rate, a fact that is well known from the theoretical literature on asset pricing.
Figure 4: Contour plots for risk-free rate and equity returns.

We now proceed with presenting the results of posterior analysis.

5. Posterior analysis using artificial data

In order to investigate the performance of the new techniques, we will perform Bayesian inference for $\theta$, using seven sets of artificial data (we refer to each data set as a “case”). The values of the parameters are chosen close to those estimated by Labadie, i.e., the risk aversion is set to 1, the discount factor to 0.95, the standard
deviation to 0.035 and the constant and the coefficient of the AR(1) process driving the endowment to 0.01 and to 0.02, respectively. All the priors are assumed to follow uniform distribution, that is, we pretend, we are completely uninformative. The estimates of the parameters are presented in figures 5 – 9. The structure of their posterior distributions is quite similar (almost same skewness and kurtosis)\(^2\).

![Prior distributions of discount factor, \( \beta \).](image)

\(^2\) Results related to sensitivity to the value of \( \varepsilon \), are reported in Appendix B of this paper.
Figure 6: Prior distributions of relative risk aversion, $\gamma$.

Figure 7: Prior distributions of standard deviation, $\sigma$. 

Endowment, in the present model, can be measured using aggregate variables like consumption, GNP or dividends, see for example Labadie (1989). This could be avoided in a model with production, but as noted in Mehra and Prescott (1985) production does not, by itself, solve the equity premium puzzle. In this paper, we use the annual data for per capita real consumption for the time period 1889-1977 of Grossman and Shiller (1981). The specific data set was used since comparisons with previous studies are desirable (for example Mehra and Prescott (1985), Rietz (1988) and Labadie (1989)). Some authors have used quarterly data (Kandel and Stambaugh (1990) but Mehra and Prescott (1985) report that their conclusions were not sensitive to the choice of data frequency.

Deriving posterior distributions of asset returns is important in judging the model’s ability to reproduce key statistics. What is needed is to incorporate all available information in the posterior distributions of parameters to perform asset return inference. Duffie and Singleton (1994) formalize the moment matching criterion implicit in Mehra and Prescott’s (1985) approach by using a simulated moments estimator. While extremely useful, this technique does not deliver distributions of asset returns in finite samples. The method adopted here (due to Meng (1994)) is the following: Given a sample \( \{\theta^{(r)}; s = 1, ..., R\} \) from the posterior distribution, \( p(\theta | Y) \), asset returns \( R^q(\theta) \) and \( R^f(\theta) \) in (4) and (5), are functions of interest whose evaluation is possible for each draw. Specifically, given \( \theta^{(r)} \), a set of \( S \) time series \( \{y^{(r)}_t, t = 1, ..., n\} \) is generated \( s = 1, ..., S \), asset returns \( R^q_{(s,t)}(\theta^{(r)}) \) and \( R^f_{(s,t)}(\theta^{(r)}) \) are computed for each date \( t \) and each simulation \( s \), and their time series averages:

\[
\bar{R}^q(\theta^{(r)}) = (ST)^{-1} \sum_{s=1}^{S} \sum_{t=1}^{T} R^q_{(s,t)}(\theta^{(r)})
\]

\[
\bar{R}^f(\theta^{(r)}) = (ST)^{-1} \sum_{s=1}^{S} \sum_{t=1}^{T} R^f_{(s,t)}(\theta^{(r)})
\]

are computed for each \( r = 1, ..., R \). Standard kernel density procedures can then be used to estimate the posterior distribution of average asset returns. In this case, \( T = 91 \) and \( S = 500 \). To compute stock prices, \( q_t \), the infinite sum in (5) needs to be truncated. Let
\( q_t^{(L)} \) denote the series \( q_t^{(L)} = Y_t \sum_{l=1}^{L} A_l y_t^{nl} \), i.e. stock prices computed with the infinite sum truncated at \( L \) iterations. We determine \( L \) so that \( |q_t^{(L+1)} - q_t^{(L)}| \leq \varepsilon \) for all \( t = 1, \ldots, T \), and \( \varepsilon = 10^{-3} \). This criterion is quite stringent since we are interested in absolute, not relative convergence of the series \( q_t^{(L)} \).

We will use four different prior distributions:
- \( \ln \gamma \sim N(-0.5,1) \) and \( \sigma_u \sim IG(4.77,0.19) \)
- \( \ln \gamma \sim N(-0.5,1) \) and \( \sigma_u \sim IG(2.11,0.011) \)
- \( \ln \gamma \sim N(1.445,1) \) and \( \sigma_u \sim IG(2.11,0.011) \)
- \( \ln \gamma \sim N(1.445,1) \) and \( \sigma_u \sim IG(4.77,0.19) \).

We will, additionally, examine a fifth case, where we assume that we are completely uninformative, adopting flat priors for all the parameters. Marginal posterior distributions for the structural parameters of the model, are presented in Figures 8 through 10.

![Posterior distributions of discount factor.](image)

Figure 8
Figure 8 presents the posterior distributions of discount factor, $\beta$. The distribution clearly assigns almost no probability mass to values less than about 0.7, the mean is about 0.88 with values closer to 1 being more probable.

Figure 9 shows the posterior distributions of relative risk aversion parameter, $\gamma$. These posteriors, assign almost no mass to values less than 5 and the means are close to 1.5. These posterior means provide considerable support for business cycle studies as well as for the calibration experiments of Mehra and Prescott (1985).
The marginal posterior of $\sigma_u^2$, presented in Figure 10, shows that the posterior mean of $\sigma_u^2$ is about 0.015, a value close to what is commonly used in the literature\(^3\). A general remark is that the results are not sensitive to the prior distributions, a fact that may put at rest the view that prior information could dominate the data to produce artificial conclusions about parameters or functions of interest (like asset returns). It is also to be noted that the posterior distributions of $\beta$, $\gamma$ and $\delta$ are extremely asymmetric. Consequently, it would be dangerous to rely on methods that depend on asymptotic approximations. One such method, is GMM. The frequent phenomenon of rejecting the model, when confronted with the data, along with GMM-based procedures, could be explained by the asymmetries evident in marginal posteriors of the key structural parameters.

Of course, in order to be certain, we have to provide evidence that the joint posterior predictive distribution of asset returns is consistent with what we know from

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\(^3\) The value 0.035 is commonly used. The value 0.05 could be interpreted as an upper bound when taking account of the fact that the consumption data could be measured with error (Labadie, 1989).
In figures 11 through 14, we present the joint posterior predictive distribution of average real risk-free rate and equity return computed using the simulation methodology that we described above. These posterior predictive distributions, can be used to assess whether, and to what extent, there is an asset returns puzzle. Since the sample means 1\% and 7\% are not far in the tails of the joint posterior, it turns out that the model is consistent with reality.
Figure 12

Contour plot for the case 2 of prior choices

Figure 13

Contour plot for the case 3 of prior choices
7. Conclusions

In this paper, we have shown that Bayesian inferences for the structural parameters of asset pricing models can be routinely performed using available closed form solutions for asset returns. We have proposed priors for the structural parameters and examined their implications for asset returns. The methodological contribution is twofold. First, we use the closed form solutions for asset returns to propose an econometric model in the traditional sense. Second, we use Bayesian posterior predictive distributions of asset returns to draw conclusions about what the model really implies for the observed quantities of interest. We applied the new methods to artificial data as well as in the Mehra and Prescott (1985) celebrated data set. It has been found that large equity premia and low risk-free rates are not incompatible with the posterior predictive distributions of asset returns resulting from the econometric model.
References


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Appendix A. The Jeffreys’ prior.

An interesting question concerns the form of Jeffreys’ prior for this model. Given that the Jeffreys’ prior is equal to the square root of the determinant of Fisher’s information matrix and the complexity of the model due to its high non-linearity with respect to its parameters, we will adopt some simplifications. We assume that the variance-covariance matrix is known and diagonal and that the convergence of Labadie’s algorithm is succeeded within a single iteration, that is we assume (9) and (10) apply with $M_r(\lambda) = 1$. The Jeffreys’ prior is given by the following:

$$p(\theta) \propto |I(\theta)|^{1/2},$$

where

$$I(\theta) = -E \left( \frac{\partial^2 \log p(y | \theta)}{\partial \theta \partial \theta'} \right),$$

and

$$I_{11}(\theta) = -Tm(\sigma_{11}^{-1} - (1 - \gamma)^2 \sigma_{22}^{-1}),$$

$$I_{12}(\theta) = -Tm(\sigma_{11}^{-1} + (1 - \gamma)\sigma_{22}^{-1} + \gamma^2 \sigma_{33}^{-1}),$$

$$I_{13}(\theta) = \frac{T}{\beta} (-\sigma_{22}^{-1} + \gamma \sigma_{33}^{-1}),$$

$$I_{14}(\theta) = -T((k + (1 - \gamma)\sigma_{22} + \delta m)(1 - \gamma)\sigma_{22}^{-1} + (k - \gamma \sigma_{11} + \delta m)\gamma \sigma_{33}),$$

$$I_{21}(\theta) = -Tv((1 - \gamma)^2 \sigma_{22} + \gamma^2 \sigma_{33}),$$

$$I_{22}(\theta) = -Tm(-(1 - \gamma)^2 \sigma_{22} + \sigma_{11} + \gamma^2 \sigma_{33}),$$

$$I_{23}(\theta) = -\frac{Tv}{\beta} ((1 - \gamma)\sigma_{22} + \gamma \sigma_{33}),$$

$$I_{24}(\theta) = -T((k + (1 - \gamma)\sigma_{11} + \delta m)(1 - \gamma)\sigma_{22} + (k - \gamma \sigma_{11} + \delta m)\gamma \sigma_{33},$$

$$I_{31}(\theta) = -0.5T((-k + (1 - \gamma)\sigma_{11} - \delta m)(\sigma_{11} \sigma_{22} + 2(k - \gamma \sigma_{11})\sigma_{22} + \delta m) + (-k + (1 - \gamma)\sigma_{11} - \delta m)(\sigma_{11} \sigma_{33} + 2 \sigma_{33} (k - \gamma \sigma_{11} + \delta m))).$$
\[ I_{32}(\theta) = 0.5T \beta (\sigma_{11}\sigma_{33} + 2(k - \gamma\sigma_{11} + \delta m)(\sigma_{22} + \sigma_{33})), \]
\[ I_{33}(\theta) = -0.5T(- (1 - \gamma)(\sigma_{11}\sigma_{22} + 2\sigma_{22}(k - \gamma\sigma_{11} + \delta m)) + \]
\[ + \gamma(\sigma_{11}\sigma_{33} + 2\sigma_{33}(k - \gamma\sigma_{11} + \delta m)), \]
\[ I_{34}(\theta) = -0.5T(-(1 - \gamma)(\sigma_{11}\sigma_{22} + 2\sigma_{22}(k - \gamma\sigma_{11} + \delta m))(\sigma_{22} + \sigma_{33})) + \]
\[ + \gamma m(\sigma_{11}\sigma_{33} + 2\sigma_{33}(k - \gamma\sigma_{11} + \delta m)), \]
\[ I_{41}(\theta) = -\frac{T}{\beta^2} \sigma_{22}, \]
\[ I_{42}(\theta) = -\frac{T}{\beta}((1 - \gamma)\sigma_{22} - \gamma\sigma_{33}), \]
\[ I_{43}(\theta) = -\frac{Tm}{\beta}((1 - \gamma)\sigma_{22} - \gamma\sigma_{33}), \]
\[ I_{44}(\theta) = \frac{T}{\beta}((k + (1 - \gamma)^2 \sigma_{11} + \delta m)\sigma_{22} + (k - \gamma\sigma_{11} + \delta m)\sigma_{33}), \]
where
\[ m \equiv E(\ln \lambda_t) = \frac{k}{1 - \delta}, \quad \nu \equiv E(\ln \lambda_t^2) = \frac{\sigma_{11}}{(1 - \delta)^2} + \frac{k^2}{(1 - \delta^2)}. \]

In Figure A1 below, we present the Jeffreys’ prior evaluated at the following parameters of the AR(1) process for endowment: \( k = 0.017165 \), and \( \delta = 0.018145 \).
From Figure A1, it is easy to see that the Jeffreys’ prior places considerable mass on large values of the relative risk aversion coefficient, and low values of the discount factor. Given experience with asset pricing and real business cycle models, this prior is quite unreasonable for use with actual data.
Appendix B. Sensitivity of results to value of $\varepsilon$.

The figures below show the sensitivity of the results with respect to the tolerance $\varepsilon$ used in order to truncate the infinite sum that gives the stock prices.

The results remain almost the same for the three last values of the tolerance, so setting $\varepsilon = 10^{-5}$ should be sufficient in practice.