Interest Rate Markets
with Stochastic Volatility

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OVERVIEW

★ **Main problem:** How to deal with *multiscale stochastic volatility* in interest rate markets?

★ **Objective:** Price zero-coupon bonds and bond options.


Interest Rate Markets

★ A **zero-coupon bond** with maturity $T$ is a contingent claim that guarantees the holder 1 dollar at $T$.

★ **No-arbitrage price** of a zero-coupon bond is:

\[
B(t, x; T) = E^* \left[ e^{-\int_t^T r_s ds} \cdot 1 \mid r_t = x \right].
\]

$E^*[\cdot] \equiv$ w.r.t. **Equivalent Martingale Measure** (EMM), under which the discounted prices of all traded assets are martingales.

★ Model dynamics of the short rate: **Vasicek, CIR, HJM**, others.
The Vasicek Model

- Short rate modelled as mean-reverting Ornstein-Uhlenbeck:

\[ dr_t = a(r_\infty - r_t)dt + \sigma dW_t. \]

\( \frac{1}{a} \) = time of mean reversion, \( r_\infty \) = long-term mean, \( \sigma \) = volatility.

- Pros:

  - SDE is easy to solve.
  - The process \( r_t \) is normally distributed.
  - Closed-form solution for the Bond price.

- Comment: Positive probability of negative short rate.
The Vasicek Price

⋆ Under the EMM,

\[ dr_t = a(r^* - r_t)dt + \sigma \, d\tilde{W}_t , \quad \text{where} \quad r^* = r_\infty - \frac{\lambda \sigma}{a} . \]

⋆ By Feynman-Kac, \( B_V \) satisfies:

\[
\begin{cases}
\frac{\partial B_V}{\partial t} + \frac{1}{2} \sigma^2 \frac{\partial^2 B_V}{\partial x^2} + a(r^* - x) \frac{\partial B_V}{\partial x} - x \, B_V = 0 \\
B_V(T, x; T, \sigma, r^*) = 1,
\end{cases}
\]

⋆ This PDE has closed-form solution:

\[ B_V = e^{-R_\infty \tau} + (R_\infty - x) B(\tau) - \frac{\sigma^2}{4a} B(\tau)^2 \]

\[ \tau = T - t, \quad B(\tau) = \frac{1}{a} (1 - e^{-a\tau}), \quad R_\infty = r^* - \frac{\sigma^2}{2a^2} . \]
Bond Price and Yield Curve for Vasicek Model


\[ a = 1, \quad \sigma = 0.1, \quad r^* = 0.1, \quad R_\infty = 0.095, \quad x = 0.05. \]
Why Stochastic Volatility?

★ **Historical data** of standard deviation of returns show that volatility is not constant.

★ **Distributions of returns** are not normal (fat tails).

★ **Smile effect** observed in implied volatilities. *(Yield curves with shapes that do not fit constant volatility.)*
Volatility Paths and Time Scales

**Top:** Slow scale.  **Bottom:** Fast scale.
★ **First path:** volatility is *low* (under 14%) for the first 17 years, and then it’s *high* for the rest of the time.

★ **Second path:** volatility is *high* for several months, and then *low* for a similar period. Then high again, and so on.

★ The second path exhibits *volatility clustering* (the tendency of volatility to come in bursts.) *When volatility is high it often stays high for a period, and similarly when it is low.*

★ **Burstiness** is closely related to *mean reversion.* The shorter the periods of the bursts, the more often it returns to its mean. *This is the effect we want to model.*
How Stochastic Volatility?

★ Choose \( \alpha \) large, and \( \delta \) small, and define:

\[

dr_t = a(r_\infty - r_t)dt + \sigma_t \, dW_t^0 \\
\sigma_t = f(Y_t, Z_t) \\
dY_t = \alpha (m - Y_t)dt + \beta \, dW_t^1 \quad (\alpha \text{ large}) \\
dZ_t = \delta c(Z_t)dt + \sqrt{\delta} \, g(Z_t) \, dW_t^2 \quad (\delta \text{ small})
\]

★ **Mean reversion time** of \( Y_t \) is \( 1/\alpha \). \( (\epsilon = 1/\alpha) \)

\[\longrightarrow Y_t \text{ varies on the } O(\epsilon) \text{ scale.} \quad (\epsilon \text{ small})\]

★ **Characteristic time scale** of \( Z_t \) is \( 1/\delta \).

\[\longrightarrow Z_t \text{ varies on the } O(1/\delta) \text{ scale.} \quad (1/\delta \text{ large})\]
Multiple Scale Framework

- **The $O(1)$ scale**: time-to-maturity scale.  
  \[(T)\]

- **The $O(\epsilon)$ fast scale**: mean reversion time of $Y_t$.  
  \[(\epsilon = \frac{1}{\alpha} < T)\]

- **The $O(1/\delta)$ slow scale**: characteristic time scale of $Z_t$.  
  \[(T < \frac{1}{\delta})\]
Two-Scale Volatility

Slow Scale Volatility $\sigma_t = f(Z_t)$

Fast Scale Volatility $\sigma_t = f(Y_t)$

Two-scale Volatility (fast and slow) $\sigma_t = f(Y_t, Z_t)$

Vasicek Price with Stochastic Volatility

★ By **Feyman-Kac**, the bond price $P_{t}^{\epsilon,\delta}$ is a solution of

\[
\left( \frac{1}{\epsilon} \mathcal{L}_0 + \frac{1}{\sqrt{\epsilon}} \mathcal{L}_1 + \mathcal{L}_2 + \sqrt{\delta} \mathcal{M}_1 + \delta \mathcal{M}_2 + \sqrt{\frac{\delta}{\epsilon}} \mathcal{M}_3 \right) P_{t}^{\epsilon,\delta} = 0 ,
\]

with terminal condition $P^{\epsilon,\delta}(T, x, y, z; T) = 1$.

★ $\mathcal{L}_0$ is the infinitesimal generator of the fast-scale process $Y_t$.

★ $\mathcal{L}_2$ is the Vasicek operator, $\mathcal{L}_{V(\sigma,r^*)}$. [Sing-Reg]

★ This is a fairly complicated PDE $\longrightarrow$ Asymptotic Analysis.
Asymptotic Analysis

* Asymptotic analysis constructs approximations to the solutions of PDEs that may be difficult or impossible to solve otherwise.

→ The approximation is a tool that allows us to extract the main behavior of the solution.

* Our approximation to the bond price will be:

\[
\tilde{P}^{\epsilon,\delta} = P_0 + \sqrt{\epsilon} P_{1,0} + \sqrt{\delta} P_{0,1}
\]

* Goal: Show that \( \tilde{P}^{\epsilon,\delta} \) is a "good" approximation to \( P^{\epsilon,\delta} \).
First Term of the Approximation \( (P_0) \)

* The first order approximation, \( P_0 \), is obtained from

\[
\begin{align*}
\mathcal{L}_V(\bar{\sigma}, \bar{r}^*) P_0 &= 0 \\
P_0(T, x, z) &= 1,
\end{align*}
\]

where \( \mathcal{L}_V(\bar{\sigma}, \bar{r}^*) \) is the Vasicek op. with effective parameters

\[
\bar{\sigma}^2(z) = \langle f^2 \rangle, \quad \bar{r}^*(z) = r_\infty - \frac{1}{a} \langle \lambda f \rangle.
\]

\( \langle \cdot \rangle \) = average w.r.t. invariant distribution of \( Y_t \).

* \( P_0 \) is the Vasicek price with homogenized parameters \( \bar{\sigma}^2 \) and \( \bar{r}^* \):

\[
P_0(t, x, z) = \mathbb{B}_V(\bar{\sigma}, \bar{r}^*)
\]

\[\longrightarrow\text{ Note that we have eliminated the influence of the fast scale.}\]
Other Terms of the Approximation

★ The **second term** in the approximation \((P_{1,0})\) is obtained from:

\[
\begin{cases}
\langle \mathcal{L}_2 \rangle P_{1,0} = A P_0 \\
P_{1,0}(T, x, z) = 0,
\end{cases}
\]

where \(A := \langle \mathcal{L}_1 \mathcal{L}_0^{-1}(\mathcal{L}_2 - \langle \mathcal{L}_2 \rangle) \rangle\).

★ The **third term** in the approximation \((P_{0,1})\) is obtained from:

\[
\begin{cases}
\langle \mathcal{L}_2 \rangle P_{0,1} = -\langle \mathcal{M}_1 \rangle P_0 \\
P_{0,1}(T, x, z) = 0.
\end{cases}
\]
Price Approximation

* Solving the previous systems we get:

\[
\tilde{P}^{\epsilon, \delta} = (1 + D^{\epsilon} + D^{\delta}) P_0,
\]

with

\[
D^{\epsilon}(\tau, z) = \left( \frac{V_3^\epsilon}{a^3} - \frac{V_2^\epsilon}{a^2} \right) \left( \tau - B - \frac{1}{2} aB^2 \right) - \frac{V_3^\epsilon}{3a} B^3 + \frac{V_1^\epsilon}{a} (\tau - B),
\]

\[
\longrightarrow V_3^\epsilon = \frac{\nu \sqrt{\epsilon}}{\sqrt{2}} \rho_1 \langle f \phi_y \rangle, \text{ etc.}
\]

\[
D^{\delta}(\tau, z) = \text{ depends on } V_0^\delta, V_1^\delta, m_1, m_2.
\]

* Main result: \( P^{\epsilon, \delta} - \tilde{P}^{\epsilon, \delta} \) is of \( O(\epsilon + \delta) \).

There exists \( C > 0 \) such that \(| P^{\epsilon, \delta} - \tilde{P}^{\epsilon, \delta} | \leq C(\epsilon + \delta) \).
The fast scale is responsible for the hump, while the slow scale "pulls down" the yield curve. The slow scale has a greater impact on the long end, while the fast scale affects the short end. This fact can be used to calibrate the model.
Implied Volatilities and Calibration

★ **Implied Volatility** \((I)\) is the value of the volatility that makes:

\[
\mathcal{B}_V (x, t; T, I, r^*) = P^{obs} (t, T)
\]

★ For fixed \(x, a\) and \(r^*\), we have a closed-form expression for \(I\):

\[
I = \sqrt{4a^3 \left\{ \frac{R_\infty - x}{a (1 - e^{-a \tau})} - \frac{R_\infty + \log P^{obs}}{(1 - e^{-a \tau})^2} \right\}},
\]

where \(R_\infty = r^* - \frac{\sigma^2}{2a^2}\).

★ If theoretical prices \((\sigma) = observed\) prices \(\Rightarrow I = \sigma\).

★ "Smile" effect (yield curves that don’t fit constant volatility)
The larger the value of $a$, the faster $r_t$ will achieve its invariant distribution, and therefore the less uncertainty (lower values of $I$).
Implementation

1. We show that group parameters can be reduced to

\[ \sigma^*, r^{**}, U_3^\epsilon, V_0^\delta, V_1^\delta, m_1, \]

where the leading order term is a solution of

\[ \mathcal{L}_{V(\sigma^*, r^{**})} P_0 = 0, \quad \text{with} \quad P_0(T, x, z) = 1, \]

2. For liquid bonds, we fit an affine function to the implied volatility surface (across different maturities).

3. From the estimated coefficients calculate effective parameters.

4. Use effective parameters to price other interest rate derivatives that are not so liquid (bond options, swaps, convertible bonds.)
Model Calibration (Fast and Slow Scales)

M2 = Monte Carlo bond price.  
BV = Vasicek bond price (constant vol.)  
P2 = Bond price approximation.

Price Approximation (P2), Monte Carlo Prices (M2) and Vasicek Price (BV) —-Optimal (V,α, ψ)
Model Calibration (Slow scale)

YMS = Monte Carlo.  YV = Vasicek.  Ys = Asymptotic approximation.

Yield curve cannot be fit with Vasicek (constant volatility).

Asymptotic approximation allows good fit.
Bond Options

* If underlying is a $T$-bond $\rightarrow$ bond option (expiration $T_0 < T$).

\[ P^{bo} = E^Q \left[ e^{-\int_t^{T_0} r_s ds} X \mid \mathcal{F}_t \right]. \]

* Problem: We need joint distribution of $\int_t^{T_0} r_s ds$ and $X$.

* Forward measures: Choose numeraire and obtain new measure. Derivative prices expressed as expectations w.r.t. new measure.

* $T$-bond as the numeraire $\rightarrow$ $T$-forward neutral measure.

\[ P^{bo}(t) = E^Q \left[ e^{-\int_t^{T_0} r_s ds} \cdot X \mid \mathcal{F}_t \right] = P(t, T) \cdot E^T \left[ X \mid \mathcal{F}_t \right]. \]

* For a European call option with strike $K$,

\[ P^{bo}(0) = P(0, T) \cdot \left( \mathcal{N}[d_1] - K \cdot \mathcal{N}[d_2] \right), \]
Bond Option Price with Stochastic Volatility

★ By **Feynman-Kac**, the bond option price satisfies:

\[
\begin{cases}
\mathcal{L}_V P^{bo} = 0 \\
P^{bo}(T_0, x; T) = h(x).
\end{cases}
\]  

(1)

★ Define the **approximation to the bond option price** as:

\[
\tilde{P}^{bo} = P^{bo}_0 + \sqrt{\epsilon} P^{bo}_{1,0} + \sqrt{\delta} P^{bo}_{0,1} = P^{bo}_0 + \tilde{P}^{bo}_F
\]

★ \(P^{bo}_0\) is the solution of (1) with parameters \(\bar{\sigma}\) and \(\bar{r}^*\).

★ The combined **first order correction** satisfies:

\[
\begin{cases}
\mathcal{L}_V (\bar{\sigma}, \bar{r}^*) \tilde{P}^{bo}_F = -\mathcal{H}_V^{\epsilon, \delta} P^{bo}_0 \\
\tilde{P}^{bo}_F(T_0, x, z; T) = 0,
\end{cases}
\]

with the source as in the case of the zero-coupon bond.
After carrying out a parameter reduction step, we obtain:

\[
\tilde{P}_F^{bo} = \sum_{i=0}^{3} D_i(t) \frac{\partial^i P_0^{bo}}{\partial x^i} + D_4(t) \frac{\partial P_0^{bo}}{\partial z} + D_5(t) \frac{\partial^2 P_0^{bo}}{\partial z \partial x},
\]

where \( D_0, \ldots, D_5 \) depend on the effective group parameters (already calibrated when pricing the zero-coupon bond).

**Remarks**

- **Group parameters** do not give enough information to recover the law of the process \((r_t, Y_t, Z_t)\), but give enough to price more complicated interest-rate derivatives.

- Once the group parameters \((U_3^\epsilon, V_0^\delta, V_1^\delta, m_1)\) are calibrated, they can be used to compute the prices of other derivatives that are not so liquid (options on bonds, convertible bonds, etc.)
Conclusions

1. Gives rise to singular and regular perturbation problems.

2. Reduction of parametric dependence of pricing formulas:
   \[ \rightarrow \text{Model parameters:} \]
   \[ a, r_\infty, \sigma, \alpha, \delta, m, \beta, \rho_1, \rho_2, \rho_{12}, f(\cdot), \lambda(\cdot), \gamma(\cdot), \xi(\cdot). \]
   \[ \rightarrow \text{Group parameters:} \]
   \[ a, \sigma^*, r^{**}, U_3^\epsilon, V_0^\delta, V_1^\delta, m_1. \]

3. Do not need to specify nor estimate \( f, \lambda, \gamma, \xi \).

4. Corrected prices of bond options depend on group parameters.

5. Extension to multi-dimensional case \( \rightarrow \) Credit Risk.