Perturbed Gaussian Copula

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Modeling Correlated Defaults: First Passage Model under Stochastic Volatility
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Copula

**Copula**: a joint distribution function of uniform random variables.

**Sklar’s Theorem**: for any multivariate distribution, the univariate marginal distributions and the dependence structure can be separated. The dependence structure is completely determined by the copula.

Joint distribution of uniform RVs: $F^*(u_1, \cdots, u_n)$

\[
F(x_1, \cdots, x_n) = F^*(F_X(x_1), \cdots, F_X(x_n)) \\
= \mathbb{I}P(U_1 \leq F_X(x_1), \cdots, U_n \leq F_X(x_n)) \\
= \mathbb{I}P(F_X^{-1}(U_1) \leq x_1, \cdots, F_X^{-1}(U_n) \leq x_n) \\
= \mathbb{I}P(X_1 \leq x_1, \cdots, X_n \leq x_n)
\]
Consequence: one can “borrow” the dependence structure, namely the copula, of one set of dependent random variables and exchange the marginal distributions for a totally different set of marginal distributions.

Invariance under monotonic transformation: if $g_i$ is strictly increasing for each $i$, then $(g_1(X_1), g_2(X_2), \ldots, g_n(X_n))$ have the same copula as $(X_1, X_2, \ldots, X_n)$.

Gaussian copula: Let $(Z_1, \ldots, Z_n)$ be a normal random vector with standard normal marginals and correlation matrix $R$, and $\Phi(\cdot)$ be the standard normal cumulative distribution function. Then the joint distribution function of $(\Phi(Z_1), \ldots, \Phi(Z_n))$ is called the Gaussian copula with correlation matrix $R$. 
Sampling with Copula

If \( F \) is a given one-dimensional cdf then

\[
F^{-1}(\Phi(Z_1)), \ldots, F^{-1}(\Phi(Z_n))
\]

are \( n \) random variables with marginals equal to \( F \) and correlated by the Gaussian copula.

Consider \( n = 2 \) for simplicity:

\((Z_1, Z_2)\) could be obtained as \((W^{(1)}_1, W^{(2)}_1)\) with two standard Brownian motions with correlation

\[
d\langle W^{(1)}, W^{(2)} \rangle_t = \rho dt
\]

Indeed that would be foolish in the Gaussian case BUT

\[
\rightarrow
\]
Model Setup

Motivated by stochastic volatility models, we start with a process \((X_t^{(1)}, X_t^{(2)}, Y_t)\) which follows the dynamics:

\[
\begin{align*}
\frac{dX_t^{(1)}}{dt} &= f_1(Y_t)\frac{dW_t^{(1)}}{dt} \\
\frac{dX_t^{(2)}}{dt} &= f_2(Y_t)\frac{dW_t^{(2)}}{dt} \\
\frac{dY_t}{dt} &= \frac{1}{\epsilon}(m - Y_t)dt + \frac{\nu\sqrt{2}}{\sqrt{\epsilon}}\frac{dW_t^{(Y)}}{dt}
\end{align*}
\]

where \((W_t^{(1)}, W_t^{(2)}, W_t^{(Y)})\) are correlated standard Brownian motions.

The process \(Y_t\) is a fast mean-reverting Ornstein-Uhlenbeck with invariant distribution \(\mathcal{N}(m, \nu^2)\) and rate of mean-reversion \(1/\epsilon\).
Correlation Structure

\[ d\langle W^{(1)}, W^{(2)} \rangle_t = \rho dt, \quad d\langle W^{(1)}, W^{(Y)} \rangle_t = \rho_1 Y dt, \quad d\langle W^{(2)}, W^{(Y)} \rangle_t = \rho_2 Y dt, \]

with \(|\rho|^2 \leq 1, |\rho_1 Y|^2 \leq 1, |\rho_2 Y|^2 \leq 1\) making the correlation matrix

\[
\begin{bmatrix}
1 & \rho & \rho_1 Y \\
\rho & 1 & \rho_2 Y \\
\rho_1 Y & \rho_2 Y & 1
\end{bmatrix}
\]
symmetric positive definite.

The \( f_i \)'s are real functions for \( i = 1, 2 \), and are assumed here to be bounded above and below away from 0.
For a fixed time $T > 0$, our objective is to find, for $t < T$, the joint distribution

$$\mathbb{P} \left\{ X_T^{(1)} \leq \xi_1, X_T^{(2)} \leq \xi_2 \mid X_t = x, Y_t = y \right\}$$

and the two marginal distributions

$$\mathbb{P} \left\{ X_T^{(1)} \leq \xi_1 \mid X_t = x, Y_t = y \right\}, \quad \mathbb{P} \left\{ X_T^{(2)} \leq \xi_2 \mid X_t = x, Y_t = y \right\},$$

where $X_t \equiv (X_t^{(1)}, X_t^{(2)}), \ x \equiv (x_1, x_2)$, and $\xi_1, \xi_2$ are two arbitrary numbers. Equivalently, we need to find the following three transition densities:

$$u^\epsilon \equiv \mathbb{P} \left\{ X_T^{(1)} \in d\xi_1, X_T^{(2)} \in d\xi_2 \mid X_t = x, Y_t = y \right\},$$

$$v_1^\epsilon \equiv \mathbb{P} \left\{ X_T^{(1)} \in d\xi_1 \mid X_t = x, Y_t = y \right\},$$

$$v_2^\epsilon \equiv \mathbb{P} \left\{ X_T^{(2)} \in d\xi_2 \mid X_t = x, Y_t = y \right\},$$

where we show the dependence on the small parameter $\epsilon$. 
PDE Representation

Let us consider $u^\epsilon$ first. In terms of partial differential equation (PDE), $u^\epsilon$ satisfies the following Kolmogorov backward equation

$$
\mathcal{L}^\epsilon u^\epsilon(t, x_1, x_2, y) = 0
$$

$$
u^\epsilon(T, x_1, x_2, y) = \delta(\xi_1; x_1)\delta(\xi_2; x_2)
$$

where $\delta(\xi_i; x_i)$ is the Dirac delta function of $\xi_i$ with spike at $\xi_i = x_i$ for $i = 1, 2$, and operator $\mathcal{L}^\epsilon$ has the following decomposition:

$$
\mathcal{L}^\epsilon = \frac{1}{\epsilon} \mathcal{L}_0 + \frac{1}{\sqrt{\epsilon}} \mathcal{L}_1 + \mathcal{L}_2 \\
\text{with: } \quad \mathcal{L}_0 = (m - y) \frac{\partial}{\partial y} + \nu^2 \frac{\partial^2}{\partial y^2}
$$

$$
\mathcal{L}_1 = \nu \sqrt{2} \rho_1 Y f_1(y) \frac{\partial^2}{\partial x_1 \partial y} + \nu \sqrt{2} \rho_2 Y f_2(y) \frac{\partial^2}{\partial x_2 \partial y}
$$

$$
\mathcal{L}_2 = \frac{\partial}{\partial t} + \frac{1}{2} f_1^2(y) \frac{\partial^2}{\partial x_1^2} + \frac{1}{2} f_2^2(y) \frac{\partial^2}{\partial x_2^2} + \rho f_1(y) f_2(y) \frac{\partial^2}{\partial x_1 \partial x_2}
$$
Expansion

We expand the solution $u^\epsilon$ in powers of $\sqrt{\epsilon}$:

$$u^\epsilon = u_0 + \sqrt{\epsilon} u_1 + \epsilon u_2 + \epsilon^{3/2} u_3 + \cdots$$

In the following, we will determine the first few terms appearing on the right hand side of the above expansion. Specifically, we will retain

$$\bar{u} \equiv u_0 + \sqrt{\epsilon} u_1$$

as an approximation to $u^\epsilon$.

(later we will propose another approximation in order to restore positiveness.)
Leading Order Term $u_0$

The leading order term $u_0$, which is independent of variable $y$, is characterized by:

$$\langle L_2 \rangle u_0(t, x_1, x_2) = 0,$$

$$u_0(T, x_1, x_2) = \delta(\xi_1; x_1)\delta(\xi_2; x_2),$$

where $\langle \cdot \rangle$ denotes the average with respect to the invariant distribution $\mathcal{N}(m, \nu^2)$ of $Y_t$, i.e.,

$$\langle g \rangle \equiv \int_{-\infty}^{\infty} g(y) \frac{1}{\nu \sqrt{2\pi}} \exp \left\{ -\frac{(y - m)^2}{2\nu^2} \right\} dy$$

for a general function $g$ of $y$.

We define the effective volatilities $\bar{\sigma}_1$ and $\bar{\sigma}_2$, and the effective correlation $\bar{\rho}$ by:

$$\bar{\sigma}_1 \equiv \sqrt{\langle f_1^2 \rangle}, \quad \bar{\sigma}_2 \equiv \sqrt{\langle f_2^2 \rangle}, \quad \bar{\rho} \equiv \frac{\rho \langle f_1 f_2 \rangle}{\bar{\sigma}_1 \bar{\sigma}_2}$$
The equation for \( u_0 \) becomes

\[
\frac{\partial u_0}{\partial t} + \frac{1}{2} \bar{\sigma}_1^2 \frac{\partial^2 u_0}{\partial x_1^2} + \frac{1}{2} \bar{\sigma}_2^2 \frac{\partial^2 u_0}{\partial x_2^2} + \bar{\rho} \bar{\sigma}_1 \bar{\sigma}_2 \frac{\partial^2 u_0}{\partial x_1 \partial x_2} = 0
\]

\[
u_0(T, x_1, x_2) = \delta(\xi_1; x_1) \delta(\xi_2; x_2)
\]

It can be verified that \( u_0 \) is the transition density of two correlated scaled Brownian motions with instantaneous correlation \( \bar{\rho} \) and scale factors \( \bar{\sigma}_1 \) and \( \bar{\sigma}_2 \), respectively. That is,

\[
u_0(t, x_1, x_2) = \frac{1}{2\pi\bar{\sigma}_1 \bar{\sigma}_2 (T - t) \sqrt{1 - \bar{\rho}^2}} \times
\exp \left\{ -\frac{1}{2(1 - \bar{\rho}^2)} \left[ \frac{(\xi_1 - x_1)^2}{\bar{\sigma}_1^2 (T - t)} - \frac{2\bar{\rho} (\xi_1 - x_1)(\xi_2 - x_2)}{\bar{\sigma}_1 \bar{\sigma}_2 (T - t)} + \frac{(\xi_2 - x_2)^2}{\bar{\sigma}_2^2 (T - t)} \right] \right\}
\]
**Correction Term** \( \sqrt{\epsilon} u_1 \)

The correction term \( u_1 \), which is also independent of variable \( y \), is characterized by:

\[
\langle L_2 \rangle u_1(t, x_1, x_2) = Au_0
\]

\[
u_1(T, x_1, x_2) = 0
\]

where the operator \( A \) is defined by

\[
A = \langle L_1 L_0^{-1}(L_2 - \langle L_2 \rangle) \rangle
\]

and the inverse \( L_0^{-1} \) is taken on the centered quantity \( L_2 - \langle L_2 \rangle \).
From the definition of $\mathcal{L}_2$, one has

$$\mathcal{L}_2 - \langle \mathcal{L}_2 \rangle =$$

$$\frac{1}{2}(f_1^2(y) - \langle f_1^2 \rangle)\frac{\partial^2}{\partial x_1^2} + \frac{1}{2}(f_2^2(y) - \langle f_2^2 \rangle)\frac{\partial^2}{\partial x_2^2} + \rho(f_1(y)f_2(y) - \langle f_1f_2 \rangle)\frac{\partial^2}{\partial x_1 \partial x_2}$$

Denoting by $\phi_1(y), \phi_2(y)$ and $\phi_{12}(y)$ solutions of the following Poisson equations

$$\mathcal{L}_0\phi_1(y) = f_1^2(y) - \langle f_1^2 \rangle$$
$$\mathcal{L}_0\phi_2(y) = f_2^2(y) - \langle f_2^2 \rangle$$
$$\mathcal{L}_0\phi_{12}(y) = f_1(y)f_2(y) - \langle f_1f_2 \rangle$$

the operator $\sqrt{\epsilon \mathcal{A}}$ can be written

$$\sqrt{\epsilon \mathcal{A}} = R_1 \frac{\partial^3}{\partial x_1^3} + R_2 \frac{\partial^3}{\partial x_2^3} + R_{12} \frac{\partial^3}{\partial x_1 \partial x_2^2} + R_{21} \frac{\partial^3}{\partial x_1^2 \partial x_2}$$

The constant group parameters $R_1, R_2, R_{12}, R_{21}$ are related to the model parameters by:
\[ R_1 \equiv \frac{\nu \rho_1 Y \sqrt{\epsilon}}{\sqrt{2}} \langle f_1 \phi'_1 \rangle, \quad R_2 \equiv \frac{\nu \rho_2 Y \sqrt{\epsilon}}{\sqrt{2}} \langle f_2 \phi'_2 \rangle \]

\[ R_{12} \equiv \frac{\nu \rho_1 Y \sqrt{\epsilon}}{\sqrt{2}} \langle f_1 \phi'_2 \rangle + \nu \sqrt{2 \epsilon} \rho_2 Y \langle f_2 \phi'_{12} \rangle \]

\[ R_{21} \equiv \frac{\nu \rho_2 Y \sqrt{\epsilon}}{\sqrt{2}} \langle f_2 \phi'_1 \rangle + \nu \sqrt{2 \epsilon} \rho_1 Y \langle f_1 \phi'_{12} \rangle \]

Note that they are all small of order \( \sqrt{\epsilon} \).

It can be checked directly that \( u_1 \) is given explicitly by

\[ u_1 = -(T - t) A u_0 \]

and therefore

\[ \sqrt{\epsilon} u_1 = -(T - t) \left[ R_1 \frac{\partial^3}{\partial x_1^3} + R_2 \frac{\partial^3}{\partial x_2^3} + R_{12} \frac{\partial^3}{\partial x_1 \partial x_2^2} + R_{21} \frac{\partial^3}{\partial x_1^2 \partial x_2} \right] u_0 \]

with explicit formulas for these derivatives of \( u_0 \).
Regularity Conditions for Density Functions

Since
\[ 1 = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} u_0(t, x_1, x_2; \xi_1, \xi_2) d\xi_1 d\xi_2, \]

by Lebesgue dominated convergence theorem, we then have
\[ 0 = \frac{\partial^{k_1+k_2} 1}{\partial x_1^{k_1} \partial x_2^{k_2}} = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{\partial^{k_1+k_2}}{\partial x_1^{k_1} \partial x_2^{k_2}} u_0(t, x_1, x_2; \xi_1, \xi_2) d\xi_1 d\xi_2, \]

for integers \( k_1, k_2 \geq 0 \). It follows that
\[ \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \sqrt{\epsilon} u_1(t, x_1, x_2; \xi_1, \xi_2) d\xi_1 d\xi_2 = 0, \]

and hence
\[ \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \tilde{u}(t, x_1, x_2; \xi_1, \xi_2) d\xi_1 d\xi_2 = 1, \]

where \( \tilde{u} = u_0 + \sqrt{\epsilon} u_1 \) is our approximation.
In order to guarantee that our approximated transition density function is always non-negative we seek a multiplicative perturbation of the form

\[ \tilde{u} \equiv \hat{u}_0 (1 + \tanh(\sqrt{\epsilon} \hat{u}_1)) \]

where \( \hat{u}_0 \) and \( \hat{u}_1 \) are defined such that

\[ u_0 + \sqrt{\epsilon} u_1 = \hat{u}_0 (1 + \sqrt{\epsilon} \hat{u}_1) \]

for any \( \epsilon > 0 \). It can be easily seen that this is achieved with the choice: \( \hat{u}_0 = u_0, \quad \hat{u}_1 = u_1 / u_0 \).

Now instead of using \( \bar{u} \) as our approximation for \( u^\epsilon \), we use

\[
\tilde{u} = u_0 \left[ 1 + \tanh\left( \frac{1}{u_0} \left[ R_1 \frac{\partial^3 u_0}{\partial x_1^3} + R_2 \frac{\partial^3 u_0}{\partial x_2^3} + R_{12} \frac{\partial^3 u_0}{\partial x_1 \partial x_2} + R_{21} \frac{\partial^3 u_0}{\partial x_1^2 \partial x_2} \right] \right) \right]
\]

Since \( \tanh(x) \approx x - x^3 / 3 \) the accuracy remains but we need to show that \( \tilde{u} \) is indeed a probability density function.
**Definition:** A function $g$ of $n$ variables $(x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$ is called an $n$-dimensional **even** function if

$$g(-x_1, -x_2, \ldots, -x_n) = g(x_1, x_2, \ldots, x_n)$$

for all $(x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$, and an $n$-dimensional **odd** function if

$$g(-x_1, -x_2, \ldots, -x_n) = -g(x_1, x_2, \ldots, x_n)$$

for all $(x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$.

**Proposition:** Let $g(x)$ be a probability density function on $\mathbb{R}^n$, and $\varphi(x)$ be an odd function. If $g$ is an even function, then the function $f$ defined by

$$f(x) = (1 + \tanh(\varphi(x))) g(x)$$

is also a probability density function on $\mathbb{R}^n$. 
**Proof** We need to prove that $f$ is globally non-negative and its integral over $\mathbb{R}^n$ is equal to one. Observe that $\tanh(\cdot)$ is strictly between $-1$ and 1, and this together with the non-negativity of $g$ justifies that $f$ is always non-negative. On the other hand, $\tanh(\cdot)$ is a (1-dimensional) odd function, and hence $\tanh(\varphi(x))$ is an ($n$-dimensional) odd function. Now by change of variables $y = -x$, we have

$$I \equiv \int_{\mathbb{R}^n} \tanh(\varphi(x))g(x)dx = \int_{\mathbb{R}^n} \tanh(\varphi(-y))g(-y)dy$$

$$= -\int_{\mathbb{R}^n} \tanh(\varphi(y))g(y)dy = -I,$$

which implies that $I = 0$. Therefore

$$\int_{\mathbb{R}^n} f(x)dx = \int_{\mathbb{R}^n} g(x)dx + I = 1 + 0 = 1.$$

The proof is complete. ■
Now observe that $u_0$ is a probability density function with respect to the variables $(\xi_1, \xi_2)$, and is even on $(\xi_1 - x_1, \xi_2 - x_2)$.

In addition, $\sqrt{\epsilon} u_1 / u_0$ is an odd function on $(\xi_1 - x_1, \xi_2 - x_2)$.

By the Proposition we know that our approximation $\tilde{u}$ is indeed a probability density function.
**Marginal Transition Densities**

For the marginal transition density function

\[ v_1^\varepsilon \equiv \mathbb{P}\left\{ X_T^{(1)} \in d\xi_1 \mid X_t = x, Y_t = y \right\}, \]

the above argument goes analogously, and we obtain

\[ v_1^\varepsilon \approx \bar{v}_1 \equiv p_1(t, x_1; T, \xi_1 | \bar{\sigma}_1) - (T - t) R_1 \frac{\partial^3}{\partial x_1^3} p_1(t, x_1; T, \xi_1 | \bar{\sigma}_1), \]

where \( p_1(t, x_1; T, \xi_1 | \bar{\sigma}_1) \) is the transition density of the scaled Brownian motion with scale factor \( \bar{\sigma}_1 \), that is,

\[ p_1(t, x_1; T, \xi_1 | \bar{\sigma}_1) = \frac{1}{\sqrt{2\pi(T - t)\bar{\sigma}_1}} \exp\left\{ -\frac{(\xi_1 - x_1)^2}{2\bar{\sigma}_1^2(T - t)} \right\}. \]

A straightforward calculation shows that

\[ \frac{\partial^3 p_1}{\partial x_1^3} = \left[ -\frac{3(\xi_1 - x_1)}{\sqrt{2\pi} \bar{\sigma}_1^5(T - t)^{5/2}} + \frac{(\xi_1 - x_1)^3}{\sqrt{2\pi} \bar{\sigma}_1^7(T - t)^{7/2}} \right] \exp\left\{ -\frac{(\xi_1 - x_1)^2}{2\bar{\sigma}_1^2(T - t)} \right\}. \]
Note again that
\[ \int_{-\infty}^{\infty} \bar{v}_1(t, x_1; T, \xi_1) d\xi_1 = 1. \]

To guarantee the non-negativity of the approximated density function, we, again, use instead
\[ \tilde{v}_1 \equiv p_1 \left[ 1 + \tanh \left( -(T - t) R_1 \frac{1}{p_1} \frac{\partial^3 p_1}{\partial x_1^3} \right) \right] \]
as our approximation to \( v_1^\epsilon \).

By symmetry we have
\[ v_2^\epsilon \equiv \mathbb{P} \left\{ X_T^{(2)} \in d\xi_2 \mid X_t = x, Y_t = y \right\} \]
\[ \approx \bar{v}_2 \equiv p_2(t, x_2; T, \xi_2 | \bar{\sigma}_2) - (T - t) R_2 \frac{\partial^3}{\partial x_2^3} p_2(t, x_2; T, \xi_2 | \bar{\sigma}_2) \]
\[ \approx \tilde{v}_2 \equiv p_2 \left[ 1 + \tanh \left( -(T - t) R_2 \frac{1}{p_2} \frac{\partial^3 p_2}{\partial x_2^3} \right) \right], \]
\[ p_2(t, x_2; T, \xi_2|\bar{\sigma}_2) = \frac{1}{\sqrt{2\pi(T-t)\bar{\sigma}_2}} \exp \left\{ -\frac{(\xi_2-x_2)^2}{2\bar{\sigma}_2^2(T-t)} \right\} \]

\[ \frac{\partial^3 p_2}{\partial x_2^3} = \left[ -\frac{3(\xi_2-x_2)}{\sqrt{2\pi} \bar{\sigma}_2^5(T-t)^{5/2}} + \frac{(\xi_2-x_2)^3}{\sqrt{2\pi} \bar{\sigma}_2^7(T-t)^{7/2}} \right] \exp \left\{ -\frac{(\xi_2-x_2)^2}{2\bar{\sigma}_2^2(T-t)} \right\} , \]

and \( \tilde{v}_2 \) is our approximation to \( v_2^\epsilon \).

By exactly the same argument used for \( \tilde{u} \), one can show that \( \tilde{v}_1 \) and \( \tilde{v}_2 \) are indeed probability density functions of \( \xi_1 \) and \( \xi_2 \), respectively. Furthermore, the approximation accuracies remain unchanged when switching from \( \bar{v}_1 \) to \( \tilde{v}_1 \), and from \( \bar{v}_2 \) to \( \tilde{v}_2 \).
Approximated Copula Density

Now suppose that conditional on \( \{ X_t = x, Y_t = y \} \), \((X_T^{(1)}, X_T^{(2)})\) admits the copula \( \Psi(\cdot, \cdot) \), then, by Sklar’s Theorem, its density function \( \psi(\cdot, \cdot) \) can be represented as

\[
\psi(z_1, z_2) = \frac{u^e(t, x_1, x_2, y; T, \xi_1, \xi_2)}{v_1^e(t, x_1, y; T, \xi_1) v_2^e(t, x_2, y; T, \xi_2)},
\]

where

\[
\begin{align*}
    z_1 & = \mathbb{P} \left\{ X_T^{(1)} \leq \xi_1 \left| X_t = x, Y_t = y \right. \right\}, \\
    z_2 & = \mathbb{P} \left\{ X_T^{(2)} \leq \xi_2 \left| X_t = x, Y_t = y \right. \right\}.
\end{align*}
\]

Observe that if the volatility terms \((f_1(\cdot), f_2(\cdot))\) for \((X_T^{(1)}, X_T^{(2)})\) were constant numbers, say, the process \( \{Y_t\}_{t \leq T} \) was constant or the \( f_i \)'s were both identically constant, then \( \Psi \) would be a Gaussian copula.
Using our approximations to $u^\epsilon, v_1^\epsilon$ and $v_2^\epsilon$, we have

$$\psi(\zeta_1, \zeta_2) \approx \tilde{\psi}(\zeta_1, \zeta_2) \equiv \frac{\tilde{u}(t, x_1, x_2; T, \xi_1, \xi_2)}{\tilde{v}_1(t, x_1; T, \xi_1) \tilde{v}_2(t, x_2; T, \xi_2)}$$

where

$$\zeta_1 = \int_{-\infty}^{\xi_1} \tilde{v}_1(t, x_1; T, \xi_1) d\xi_1$$

$$= \int_{-\infty}^{\xi_1} p_1(t, x_1; T, \xi_1) \left[ 1 + \tanh \left( -(T - t) R_1 \frac{1}{p_1(t, x_1; T, \xi_1)} \frac{\partial^3 p_1(t, x_1; T, \xi_1)}{\partial x_1^3} \right) \right] d\xi_1$$

$$\zeta_2 = \int_{-\infty}^{\xi_2} \tilde{v}_2(t, x_2; T, \xi_2) d\xi_2$$

$$= \int_{-\infty}^{\xi_2} p_2(t, x_2; T, \xi_2) \left[ 1 + \tanh \left( -(T - t) R_2 \frac{1}{p_2(t, x_2; T, \xi_2)} \frac{\partial^3 p_2(t, x_2; T, \xi_2)}{\partial x_2^3} \right) \right] d\xi_2$$

The function $\tilde{u}$ is our approximation, and the marginals $(p_1, p_2)$ and their derivatives $\frac{\partial^3 p_1}{\partial x_1^3}, \frac{\partial^3 p_2}{\partial x_2^3}$ are given explicitly.

Before justifying that $\tilde{\psi}$ is a probability density function defined on the unit square $[0, 1]^2$, we need the following proposition.
**Proposition:** Suppose the function $\Theta(x_1, x_2, \ldots, x_n)$ is an $n$-dimensional probability density function on $\mathbb{R}^n$ for $n \geq 2$, and $h_1(x_1), h_2(x_2), \ldots, h_n(x_n)$ are 1-dimensional strictly positive probability density functions. Then the function $c$ defined on the unit hyper-square $[0, 1]^n$ by

$$c(z_1, z_2, \ldots, z_n) = \frac{\Theta(x_1, x_2, \ldots, x_n)}{\prod_{i=1}^{n} h_i(x_i)}$$

with $z_i \in [0, 1]$ given by

$$z_i = \int_{-\infty}^{x_i} h_i(y_i) dy_i$$

is a probability density function on $[0, 1]^n$. Furthermore, $c$ is a copula density function if and only if $h_1(x_1), h_2(x_2), \ldots, h_n(x_n)$ are the marginal density functions of $\Theta(x_1, x_2, \ldots, x_n)$, meaning that

$$h_i(x_i) = \int_{\mathbb{R}^{n-1}} \Theta(x_1, x_2, \ldots, x_n) dx_1 dx_2 \cdots dx_{i-1} dx_{i+1} \cdots dx_n, \ i = 1, 2, \ldots, n$$
Proof  Let $H_i$ be the cumulative distribution function of $h_i$. Then $H_i$ is strictly increasing, implying the existence of its inverse function, and

$$z_i = H_i(x_i), \quad \text{or equivalently,} \quad x_i = H_i^{-1}(z_i)$$

for each $i$. Since $\Theta$ is non-negative, and $h_i$’s are strictly positive, the function $c$ is non-negative. On the other hand,

$$\int_{[0,1]^n} c(z_1, z_2, \ldots, z_n) dz_1 dz_2 \cdots dz_n$$

$$= \int_{\mathbb{R}^n} c(H_1(x_1), H_2(x_2), \ldots, H_n(x_n)) \prod_{i=1}^n h_i(x_i) \, dx_1 dx_2 \cdots dx_n$$

$$= \int_{\mathbb{R}^n} \Theta(x_1, x_2, \ldots, x_n) \, dx_1 dx_2 \cdots dx_n = 1.$$

Therefore $c(z_1, z_2, \ldots, z_n)$ is a probability density function on $[0,1]^n$. 
Now if the additional condition is satisfied, then we have

$$\int_{[0,1]^{n-1}} c(z_1, z_2, \ldots, z_n) dz_2 \cdots dz_n$$

$$= \int_{\mathbb{R}^{n-1}} c(z_1, H_2(x_2), \ldots, H_n(x_n)) \prod_{i=2}^{n} h_i(x_i) \, dx_2 \cdots dx_n$$

$$= \frac{1}{h_1(x_1)} \int_{\mathbb{R}^{n-1}} \Theta(x_1, x_2, \ldots, x_n) \, dx_2 \cdots dx_n = 1.$$

This is to say that the marginal density function for the variable $z_1$ is one, and hence the marginal distribution for the variable $z_1$ is uniform. Similarly, we can show that the marginal distributions for the variables $z_2, \ldots, z_n$ are also uniform. By definition of copula, we know that function $c$ is a copula density function. The converse can be obtained by reversing the above argument. The proof is complete. ■
Now from definition of $\tilde{\psi}$, by combining the fact that $\tilde{u}$, $\tilde{v}_1$ and $\tilde{v}_2$ are all probability density functions, one can see that $\tilde{\psi}$ is a density function on $[0, 1]^2$ by applying the Proposition. However, $\tilde{\psi}$ is not a copula density function in general, because the additional condition required in the Proposition is not satisfied in general in our case, and hence $\tilde{\Psi}$, the “copula” corresponding to density function $\tilde{\psi}$, is not an exact copula in general.

Asymptotically, when $\epsilon$ goes to 0, for fixed $(t, x_1, x_2)$, the density $\tilde{\psi}$ converges to

$$
\phi(z_1, z_2) \equiv \frac{u_0(t, x_1, x_2; T, \xi_1, \xi_2)}{p_1(t, x_1; T, \xi_1) \ p_2(t, x_2; T, \xi_2)}
$$

with

$$
z_i = \int_{-\infty}^{\xi_i} p_i(t, x_i; T, \xi_i) \, d\xi_i = N \left( \frac{\xi_i - x_i}{\tilde{\sigma}_i \sqrt{T - t}} \right)
$$

for $i = 1, 2$, where $N(\cdot)$ denotes the univariate standard normal cdf.
One should observe that $\phi(\cdot, \cdot)$ is the two-dimensional Gaussian copula density function with correlation parameter $\bar{\rho}$, and that it depends only on the parameter $\bar{\rho}$, independent of any other variables/parameters, including $x_1, x_2, t, T, \bar{\sigma}_1, \bar{\sigma}_2$, etc.

As a consequence, $\tilde{\Psi}$ converges to the Gaussian copula $\Phi$ with correlation parameter $\bar{\rho}$. Since the method used in this paper is a perturbation method, we call $\tilde{\Psi}$ a perturbed Gaussian copula.
Figure 1: Perturbed Gaussian densities
Figure 2: Gaussian copula and perturbed Gaussian copula densities
Throughout the computation, we used the following parameters:

\[ R_1 = 0.02, \quad R_2 = 0.02, \quad R_{12} = 0.03, \]
\[ R_{21} = 0.03, \quad \bar{\rho} = 0.5, \quad T - t = 1, \]
\[ \bar{\sigma}_1 = 0.5, \quad \bar{\sigma}_2 = 0.5, \quad x_1 = 0, \quad x_2 = 0. \]
In summary, based on a **stochastic volatility model**, we derived an approximate copula function by way of singular perturbation that was introduced by Fouque, Papanicolaou and Sircar (2000). During the approximation, however, in order to make the candidate probability density functions globally non-negative, instead of directly using the obtained perturbation result, we introduced a multiplicative modification, namely the $1 + \tanh(\cdot)$ form. It turns out that this modification is both necessary (to restore positiveness) and sufficient to guarantee the resulting functions to be density functions.

Finally the resulting approximate copula — the so-called **perturbed Gaussian copula** — has a very desirable property compared to standard Gaussian copula: **tail dependence** at point $(0, 0)$.