The Skewed $t$ Distribution for Portfolio Credit Risk

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Outline

Credit Default Swaps

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- Normal Mean-Variance Mixture
- GH and Skewed $t$ distributions
- Calibration using the EM algorithm

Tail Dependence

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- Intensity-based Pricing
- Calibrating Default Intensity

Basket CDS
- Trigger variable approach to default times
- Pricing $k$-th to default
- Comparing copula vs distribution approach to calibration
Single Name CDS

CDS is an insurance contract against bond default. Buyer pays to the seller a periodic payment as a fraction $q$ of the nominal par value $M$ until

- the maturity $T$ of the contract, or
- the bond defaults at time $\tau < T$.

In exchange, buyer has the right to sell the bond for par if and when default occurs before time $T$. 
Basket CDS

A Basket CDS is written on a portfolio of $n$ bonds (or firms). Buyer pays to the seller a periodic payment as a fraction $q$ of the nominal value $M$ of the contract until

- the maturity $T$ of the contract, or
- the $k$-th default occurs at time $\tau^k < T$, $1 \leq k \leq n$.

Here, $k$ is a part of the contract = level of seniority.

In exchange, the buyer receives the right to sell the portfolio for $M$ if and when the $k$-th default occurs before time $T$.

What is the fair value of $q$ for this contract?

Function of the dependence structure of the $n$ default times: Default Contagion.
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Normal Mean-Variance Mixture

**Definition**
Let \( \mu \) and \( \gamma \) be parameter vectors in \( \mathbb{R}^d \), and \( \Sigma \) a \( d \times d \) real positive semidefinite matrix. The \( d \)-dimensional random variable \( X \) has a multivariate normal mean-variance mixture distribution if

\[
X \overset{d}{=} \mu + W\gamma + \sqrt{W}Z,
\]

where

1. \( Z \sim N_d(0, \Sigma) \), the multivariate normal with mean \( 0 \) and covariance \( \Sigma \), and
2. \( W \) is a real non-negative random variable independent of \( Z \).
Normal Mean-Variance Mixture features

From the definition,

\[ \mathbf{X} \mid W \sim N_d(\mu + W\gamma, W\Sigma). \]

The following moment formulas follow easily from the mixture definition:

\[ E(\mathbf{X}) = \mu + E(W)\gamma, \]
\[ COV(\mathbf{X}) = E(W)\Sigma + var(W)\gamma\gamma', \]

when the mixture variable \( W \) has finite variance \( var(W) \).
Generalized Hyperbolic Distributions

Definition
If $W$ is Generalized Inverse Gaussian (GIG), we say $X$ has a Multivariate Generalized Hyperbolic Distribution.


$X$ has a skewed $t$ distribution when $W$ has a certain Inverse Gamma distribution (special case of GIG)
Skewed t distribution

Definition
The positive random variable $X$ has an inverse gamma distribution, $X \sim \text{InverseGamma}(\alpha, \beta)$, $\alpha, \beta > 0$, if its probability density function is

$$f(x) = \beta^{\alpha} x^{-\alpha-1} e^{-\beta/x} / \Gamma(\alpha), \quad x > 0,$$

where $\Gamma$ is the usual gamma function.
Skewed $t$ distribution

**Definition**

The $d$-dimensional random variable $X$ has a *skewed $t$ distribution* $X \sim T_d(\nu, \mu, \Sigma, \gamma)$, if

$$X \overset{d}{=} \mu + W\gamma + \sqrt{W}Z,$$

where

1. $Z \sim N_d(0, \Sigma)$, the multivariate normal with mean $0$ and covariance $\Sigma$, and
2. $W \sim \text{InverseGamma}(\nu/2, \nu/2)$, independent of $Z$. 
Skewed $t$

The density function of $X \sim T_d(\nu, \mu, \Sigma, \gamma)$ can be written explicitly in terms of the modified Bessel function of the third kind.

The moments are:

$$E(X) = \mu + \gamma \frac{\nu}{\nu - 2},$$

$$COV(X) = \frac{\nu}{\nu - 2} \Sigma + \gamma \gamma' \frac{2\nu^2}{(\nu - 2)^2(\nu - 4)},$$

where the covariance matrix is only defined when $\nu > 4$, and the expectation only when $\nu > 2$. 
The (Student’s) $t$ distribution is obtained by setting the skewness parameter $\gamma = 0$:

$$X \stackrel{d}{=} \mu + \sqrt{W}Z.$$ 

$X \sim T_d(\nu, \mu, \Sigma, 0)$ has the joint density function

$$f(x) = c(\nu, d, \Sigma)(1 + \frac{\rho(x)}{\nu})^{-\frac{\nu+d}{2}}.$$ 

Where

$$\rho(x) = (x - \mu)'\Sigma^{-1}(x - \mu).$$

The mean and covariance are

$$E(X) = \mu, \quad COV(X) = \frac{\nu}{\nu - 2}\Sigma.$$
Suppose we have i.i.d. data $x_1, \ldots, x_n \in \mathbb{R}^d$ that we want to fit to a Skewed $t$ distribution. We seek parameters $\theta = (\nu, \mu, \Sigma, \gamma)$ to maximize the log likelihood

$$\log L(\theta; x_1, x_2, \ldots, x_n) = \sum_{i=1}^{n} \log f(x_i; \theta),$$

where $f(\cdot; \theta)$ denotes the Skewed $t$ density function.
EM algorithm

Idea: suppose the latent variables $w_1, \ldots, w_n$ were observable. We have the augmented log-likelihood function

$$\log \tilde{L}(\theta; x_1, \ldots, x_n, w_1, \ldots, w_n) = \sum_{i=1}^{n} \log f_{X|W}(x_i, w_i; \theta),$$

$$= \sum_{i=1}^{n} \log f_{X|W}(x_i|w_i; \mu, \Sigma, \gamma) + \sum_{i=1}^{n} \log h_W(w_i; \nu)$$

where $f_{X|W}(\cdot|w; \mu, \Sigma, \gamma)$ is the conditional normal $N(\mu + w \gamma, w \Sigma)$ and $h_W(\cdot; \nu)$ is the density of $InverseGamma(\nu/2, \nu/2)$. 
### E step and M step

Iterative approach to finding likelihood maximizing $\theta$:
Suppose we have a step $k$ parameter estimate $\theta^{[k]}$

- **E-step**: compute an objective function

$$Q(\theta; \theta^{[k]}) = E(\log \tilde{L}(\theta; x_1, \ldots, x_n, W_1, \ldots, W_n)|x_1, \ldots, x_n; \theta^{[k]})$$

- **M-step**: Maximize $Q$ to find $\theta^{[k+1]}$.

This requires formulas for $E(W_i|x_i, \theta^{[k]})$, etc.
Tail dependence

Definition

Tail Dependence Coefficient ($TDC$). Let $(X_1, X_2)$ be continuous with marginal distribution functions $F_1$ and $F_2$.

$$
\lambda_U = \lim_{u \uparrow 1} P[X_2 > F_2^{-1}(u) | X_1 > F_1^{-1}(u)],
$$

Joe (1997) $\lambda$ in terms of copula $C$:

$$
\lambda_U = \lim_{u \uparrow 1} \frac{[1 - 2u + C(u, u)]}{1 - u},
$$
Tail dependence

For elliptical copulas, $\lambda_U = \lambda_L$, denoted by $\lambda$. $\lambda = 0$ for a Gaussian copula. For a $t$ copula, Embrechts, Lindskog, McNeil (2001):

$$\lambda = 2 - 2T_\nu \left( \sqrt{\nu} \cdot \frac{\sqrt{1 - \rho}}{\sqrt{1 + \rho}} \right),$$

where $\rho$ is the Pearson correlation coefficient and $T_\nu$ is the univariate standardized Student’s $t$ distribution function.
Gaussian vs $t$ copula tail dependence

Figure: 1000 samples of Gaussian and $t$ copula with Kendall's $\tau = 0.5$. There are more points in both corners for $t$ copula.
Reduced form credit pricing

Let

- \( \tau \) be the default time of a firm
- \( H_t = l_{\tau \leq t} \) the default process
- \( \mathcal{H}_t = \sigma(H_s : s \leq t) \) the default time filtration
- \( F(t) = P(\tau \leq t) \) the distribution function of \( \tau \) – assume absolutely continuous
- \( \lambda(t) \) the default intensity

\[
F(t) = 1 - e^{-\int_0^t \lambda(u) \, du}
\]

- pdf of \( \tau \): \( f(t) = \lambda(t) e^{-\int_0^t \lambda(u) \, du} \)
Pricing formula

If a defaultable zero-coupon bond pays $C$ at maturity $T$ if no default, or $h(\tau)$ if default at time $\tau < T$, the time-$t$ present value of payoff is

$$Y_t = I_{\{t<\tau\leq T\}} h(\tau)e^{-\int_t^\tau r(u)du} + I_{\{\tau>T\}} Ce^{-\int_t^T r(u)du}.$$
When the only information is in $\mathcal{H}_t$, and $r(t)$ is a deterministic interest rate, we have

**Theorem (Rutkowski (1999))**

Assume that $t \leq T$, and $Y_t$ is defined as above. Then

$$E(Y_t|\mathcal{H}_t) = \mathbb{I}_{\{\tau > t\}} \left( \int_t^T h(u)\lambda(u)e^{-\int_t^u \hat{r}(v)dv} du + Ce^{-\int_t^T \hat{r}(u)du} \right),$$

where $\hat{r}(v) = r(v) + \lambda(v)$. 


Valuation of CDS

Assume deterministic interest rate, nominal value $M$, fixed recovery rate $R$, annual payments of $Mq$ until default.

- $PL = \text{present value of periodic payments}$
- $AP = \text{present value of accrued payment from last payment to default time}$
- $DL = \text{present value of the net payoff to buyer at default}$
- $PL(q) + AP(q) = DL$ determines price $q$. 
Leg formulas

If payments are at times \( t_1, \ldots, t_n \), and \( B(t, T) \) is the price of a \( T \)-bond at time \( t \),

-\[
    PL = Mq \sum_{i=1}^{n} B(0, t_i) e^{-\int_{0}^{t_i} \lambda(u) du}
\]

-\[
    AP = Mq \sum_{i=1}^{n} \int_{t_{i-1}}^{t_i} \frac{u - t_{i-1}}{t_i - t_{i-1}} B(0, u) \lambda(u) e^{-\int_{0}^{u} \lambda(s) ds} du
\]

-\[
    DL = M(1 - R) \int_{0}^{T} B(0, u) \lambda(u) e^{-\int_{0}^{u} \lambda(s) ds} du
\]
Illustration: Calibration of default Intensity

Here are default price spread quotes for five companies on 07/02/2004 (data from GFI) – annualized payment in basis points per dollar of nominal value.

<table>
<thead>
<tr>
<th>Company</th>
<th>Year 1</th>
<th>Year 2</th>
<th>Year 3</th>
<th>Year 4</th>
<th>Year 5</th>
</tr>
</thead>
<tbody>
<tr>
<td>AT&amp;T</td>
<td>144</td>
<td>144</td>
<td>208</td>
<td>272</td>
<td>330</td>
</tr>
<tr>
<td>Bell South</td>
<td>12</td>
<td>18</td>
<td>24</td>
<td>33</td>
<td>43</td>
</tr>
<tr>
<td>Century Tel</td>
<td>59</td>
<td>76</td>
<td>92</td>
<td>108</td>
<td>136</td>
</tr>
<tr>
<td>SBC</td>
<td>15</td>
<td>23</td>
<td>31</td>
<td>39</td>
<td>47.5</td>
</tr>
<tr>
<td>Sprint</td>
<td>57</td>
<td>61</td>
<td>66</td>
<td>83</td>
<td>100</td>
</tr>
</tbody>
</table>

Table: Credit default swap mid price quote, where year1, · · ·, year5 refer to maturities.
Calibration of default Intensity

It is usually assumed the default intensity is a step function, step size 1 year:

\[
\lambda(t) = \sum_{i=1}^{5} c_i I_{[T_{i-1}, T_i]}(t)
\]

Compute \(c_1\) from the 1 year CDS price, etc.
**Calibrated intensity values**

Assuming $R = 0.4$, interest rate $r = 0.045$, semiannual payments:

<table>
<thead>
<tr>
<th>Company</th>
<th>Year 1</th>
<th>Year 2</th>
<th>Year 3</th>
<th>Year 4</th>
<th>Year 5</th>
</tr>
</thead>
<tbody>
<tr>
<td>AT&amp;T</td>
<td>0.0237</td>
<td>0.0237</td>
<td>0.0599</td>
<td>0.0893</td>
<td>0.1198</td>
</tr>
<tr>
<td>Bell South</td>
<td>0.0020</td>
<td>0.0040</td>
<td>0.0061</td>
<td>0.0105</td>
<td>0.0149</td>
</tr>
<tr>
<td>Century Tel</td>
<td>0.0097</td>
<td>0.0155</td>
<td>0.0210</td>
<td>0.0271</td>
<td>0.0469</td>
</tr>
<tr>
<td>SBC</td>
<td>0.0025</td>
<td>0.0052</td>
<td>0.0080</td>
<td>0.0109</td>
<td>0.0144</td>
</tr>
<tr>
<td>Sprint</td>
<td>0.0094</td>
<td>0.0108</td>
<td>0.0127</td>
<td>0.0235</td>
<td>0.0304</td>
</tr>
</tbody>
</table>

**Table:** Calibrated default intensity
Basket CDS: trigger variable approach

Schönbucher and Schubert (2001)
Assume we have $d$ firms. For each firm $1 \leq i \leq d$ we need:

1. The default intensity $\lambda^i(t)$: a deterministic step function.

2. The survival function $S^i(t) := \exp \left(- \int_0^t \lambda^i(u) \, du \right)$.

3. The default trigger variables $U^i$: uniform random variables on $[0, 1]$. The $d$-dimensional vector $U = (U_1, U_2, \cdots, U_d)$ is distributed according to a $d$-dimensional copula $C$.

4. The default time $\tau^i$ of firm $i$, where $i = 1, \cdots, d$,

$$\tau^i := \inf\{ t : S^i(t) \leq 1 - U^i \}.$$

Since $\tau^i$ is an increasing function of $U^i$, the copula $C$ of $U$ is equal to the copula of $\tau$. 
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Since $\tau_i$ is an increasing function of $U_i$, the copula $C$ of $U$ is equal to the copula of $\tau$. 
We need to calibrate:

- $\lambda_i, i = 1, \ldots, d$, and
- the default time copula $C$.

$C$ is difficult: default data is sparse.

Common market practice is to use asset price correlations as a proxy for default time correlations, which is suggested by a Merton-style threshold view of defaults times.

What copula family? $t$ or skewed $t$ incorporate default contagion through tail dependence.
Assume a 5 year basket CDS on the same 5 firms ATT&T, Bell South, CenTel, SBC, Sprint.

Summary of method:

1. Calibrate the copula $C$, either directly (standard) or implicitly via calibration of the full distribution
2. Separately, calibrate the intensities $\lambda_i$ from single name CDS spread quotes, and so get the survival functions $S_i(t)$.
3. Using $C$, find an approximate unconditional distribution of $\tau^k$, the $k$-th default time via Monte Carlo simulation
4. Use the resulting distribution of $\tau^k$ to solve for the basket CDS spread price $q$ from

$$PL(q) + AP(q) = DL$$
Calibration of Copula vs Distribution

We take adjusted daily close prices 07/02/98 to 07/02/2004. COPULA approach:

- Use the empirical distributions for marginals to transform prices to uniform variates.
- For fixed degree of freedom $\nu$, calibrating the $t$ copula is fast (5 seconds): Di Clemente and Romano (2003), Demarta and McNeil (2005), Galiani (2003)
- No good method to calibrate $\nu$ except by direct search: looping $\nu$ from 2.001 to 20 step 0.01 takes 2.5 hours (result: $\nu = 7.40$)
- We weren’t able to calibrate the skewed $t$ copula
Calibration of Copula vs Distribution

DISTRIBUTION approach:

- We have the EM algorithm at our disposal
- Less than one minute to calibrate full $t$ or skewed $t$ distribution
- Degree of freedom $\nu = 4.31$
- Log likelihoods are almost the same: $18420.58$ (skewed $t$) $18420.20$ ($t$)
- Bonus: no extra requirement to assume a form for the marginals; copula itself not required
### Computed spread price comparisons

<table>
<thead>
<tr>
<th>Model</th>
<th>FTD</th>
<th>2TD</th>
<th>3TD</th>
<th>4TD</th>
<th>LTD</th>
</tr>
</thead>
<tbody>
<tr>
<td>Gaussian copula</td>
<td>525.6</td>
<td>141.7</td>
<td>40.4</td>
<td>10.9</td>
<td>2.2</td>
</tr>
<tr>
<td>(t) copula</td>
<td>506.1</td>
<td>143.2</td>
<td>46.9</td>
<td>15.1</td>
<td>3.9</td>
</tr>
<tr>
<td>(t) distribution</td>
<td>498.4</td>
<td>143.2</td>
<td>48.7</td>
<td>16.8</td>
<td>4.5</td>
</tr>
<tr>
<td>Skewed (t) distribution</td>
<td>499.5</td>
<td>143.9</td>
<td>49.3</td>
<td>16.8</td>
<td>4.5</td>
</tr>
</tbody>
</table>

**Table:** Spread price for k-th to default using different models
Summary

- $t$ and skewed $t$ distributions show good performance modeling stock prices
- the $t$ copula is commonly used for basket CDS, but requires separate marginals calibration and is slow
- the EM algorithm is a fast method for calibrating marginals and copula all at once, and allows us to also calibrate skewness
- Open question is how best to proxy default times