Two-Dimensional Markovian Model for Dynamics of Aggregate Credit Loss

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November 4, 2006

Advances in Econometrics, Baton Rouge, 2006
Outline

1. Motivation for a low-dimensional model of aggregate portfolio loss with the ability to fit the full surface of loss distribution
2. Calibration of stochastic intensity via local intensity
3. Calibration to the market of CDO tranches
4. Application to cancelable CDO tranches
Aggregate loss modeling

Direct modeling of the aggregate loss process $L_t$

Advantages:

- Hiding some of unnecessary complexity of single name constituents of the basket
- Easier construction of more tractable models with dynamic evolution

Disadvantages:

- Difficulty in getting access to single name hedges
- Difficulty in modeling of multiple portfolios and cashflow CDO waterfalls
“Vanilla” instruments of the market

Single tranche CDOs are synthetic derivatives of the aggregate loss due to defaults in a portfolio of credit names

\[ L_t = \sum_{\text{assets}} 1^{\text{“a” defaulted by time } t} \text{LGD(“a”)} \]

Each single tranche CDO can be replicated as a portfolio of stop-loss options

\[ \text{Tranche} = \sum_{i} \alpha_i \left( (L_{Ti} - k)^+ - (L_{Ti} - K)^+ \right) \]

\(k\) and \(K\) are the attachment points)
Surface of loss distribution

All “static” information about the distribution of loss is contained in the univariate marginals of the loss distribution,

\[ P[L_t \leq L] = P(L, t). \]

No-arbitrage conditions:

\[ t_1 < t_2 \Rightarrow P(L, t_1) \geq P(L, t_2), \]
\[ L_1 < L_2 \Rightarrow P(L_1, t) \leq P(L_2, t). \]

Prices of all tranches are fully defined by \( P(L, t) \) but additional assumptions are needed to imply \( P(L, t) \) from a handful of traded tranches.

With increasing number of liquid tranches, the ability to match the entire loss surface \( P(L, t) \) becomes a modeling requirement.
Survey of approaches to aggregate loss modeling

- Generic framework for arbitrage free evolution of loss: Sidenius et al (2005) and Schönbucher (2005); complete surface of loss is fitted by construction, but there are no fully explicit realizations of the loss process; implementation requires large-scale Monte Carlo simulations

- Specific models derived from Poisson process and its generalizations: Giesecke and Goldberg (2005), Errais et al (2006), Brigo et al. (2006); loss surface cannot be fitted; modeling clings to exact solutions based on Fourier transform technique

Default correlations

Basket of $N$ names, loss-given-default $h$, single name default probability $\pi_d(t)$.

**Independent defaults**

\[
P(L, t) = \sum_{n=0}^{L/h} p(n, t) = \sum_{n=0}^{L/h} \binom{N}{n} \pi_d(t)^n (1 - \pi_d(t))^{N-n}
\]

**Factor models**

\[
P(L, t) = \sum_{n=0}^{L/h} p(n, t) = \sum_{n=0}^{L/h} \binom{N}{n} \int \pi_d(t|X)^n (1 - \pi_d(t|X))^{N-n} dP(X)
\]

**Gaussian copula**

\[
\pi_d(t|X) = \mathcal{N} \left( \frac{\mathcal{N}^{-1}(\pi_d(t)) - \sqrt{\rho}X}{\sqrt{1-\rho}} \right), \quad dP(X) = n(X) dX
\]
Base correlations skew

Base correlations model:

In computing $E[(L_T - k)^+] - E[(L_T - K)^+]$ do the first term using Gaussian copula with $\rho_1 = \rho(k)$ and the second term using another Gaussian copula with $\rho_2 = \rho(K)$, $\rho_1 \neq \rho_2$

Figure 1: Base correlations skew for iTraxx 5y (38) on Oct 12, 2005
Getting correlation skew from conditional Poisson processes is not straightforward

Deterministic intensity $\lambda(t) \Rightarrow$ almost no correlations (no default clustering).

Affine stochastic intensity $\lambda(t) \Rightarrow$ flat correlations, almost no skew without very strong diffusion or jump terms (linear default clustering).

Adding additional clustering does not necessarily help, for example Hawkes process model of Errais, Giesecke, and Goldberg (2005)

$$d\lambda_t = \kappa(\rho(t) - \lambda_t)dt + \delta dL_t$$

still leads to fairly flat skews.
2D Markovian Model

\[ d\lambda_t = \kappa(\rho(L_t, t) - \lambda_t)dt + \sigma\sqrt{\lambda_t}dW_t \]

We now have sufficient freedom to calibrate the entire surface of loss distribution by adjusting the free function \( \rho(L, t) \) for any volatility \( \sigma \). (Other forms of the diffusion term are possible, also we could add jumps.)

Calibration of \( \rho(L, t) \) to the surface of loss (and the tranches) can be done without simulation.

Instruments dependent on the dynamics can be computed either by a forward simulation or by backward induction.
Calibration procedure

Tranche prices

⇓

Loss distribution $P(L, t)$

⇓

Local intensity $\Lambda(L, t)$

⇓

Stochastic intensity with “reversion” $\rho(L, t)$
Calibration problem

Forward Kolmogorov equation for the joint density $p(\lambda, L, t)$,

$$
\frac{\partial p(\lambda, L, t)}{\partial t} = \left( -\kappa \frac{\partial}{\partial \lambda} (\rho(L, t) - \lambda) + \frac{1}{2} \frac{\partial^2}{\partial \lambda^2} \sigma^2 \lambda \right) p(\lambda, L, t)
$$

$$
+ \lambda (p(\lambda, L - h, t) - p(\lambda, L, t)) .
$$

Boundary and initial conditions

$$
p(0, L, t) \equiv 0, \quad \lambda \frac{\partial}{\partial \lambda} p(\lambda, L, t)|_{\lambda=+0} = 0, \quad p(\lambda, L, 0) = p_0(\lambda) \cdot 1_{L=0}
$$

Calibration constraint

$$
P(L, t) = \int_0^\infty p(\lambda, L, t) d\lambda.
$$
Digression: local volatility

Stochastic volatility model ($\alpha_t, \beta_t$ – adapted processes)

$$dX_t = \alpha_t dt + \beta_t dW_t,$$

Local volatility model

$$dY_t = a(Y_t, t) dt + b(Y_t, t) dW_t.$$

Gyöngy (1986) - Dupire (1997) lemma: one-dimensional marginal distributions (and therefore European options) for $X_t$ and $Y_t$ are identical provided $X_0 = Y_0$ and

$$a(x, t) = E[\alpha_t | X_t = x], \quad b^2(x, t) = E[\beta_t^2 | X_t = x].$$

Dupire (1994) gave a formula for $b(x, t)$ in terms of European options.
Gyöngy-Dupire for counting processes

$N_t$ has adapted stochastic intensity $\lambda_t$

$M_t$ has local intensity $\Lambda(M, t)$

One-dimensional marginal distributions of $N_t$ and $M_t$ are identical provided $N_0 = M_0$ and

$$\Lambda(M, t) = \text{E}[\lambda_t | N_t = M].$$

We used a special case of this identity,

$$\Lambda(L, t) = \frac{\int_0^\infty \lambda p(\lambda, L, t) d\lambda}{\int_0^\infty p(\lambda, L, t) d\lambda}.$$

The counterpart of Dupire’s formula is

$$\Lambda(L, t) = -\frac{1}{P(L, t)} \frac{\partial P[L_t \leq L]}{\partial t}.$$
Local intensity model (a.k.a. 1-step Markov chain, implied loss model Van der Voort, 2006)

Forward Kolmogorov equation for loss distribution

\[
\frac{\partial P(L, t)}{\partial t} = \Lambda(L - h, t)P(L - h, t) - \Lambda(L, t)P(L, t).
\]

Local intensity \( \Lambda(L, t) \) is related to the density \( p(\lambda, L, t) \),

\[
\Lambda(L, t)P(L, t) = \int_0^\infty \lambda p(\lambda, L, t)d\lambda, \quad (*)
\]

and is easily found from the loss distribution

\[
\Lambda(K, t) = -\frac{1}{P(K, t)} \frac{\partial}{\partial t} \sum_{n=0}^{[K/h]} P(nh, t).
\]

Taking time derivative of (*) and using Kolmogorov equations for \( p(\lambda, L, t) \) and \( P(L, t) \), we can isolate \( \rho(L, t) \),
Numerical solution for $\rho(L, t)$

Instead of integral constraint on $p$ we have an equation for $\rho$,

\[
\rho(L, t) = \Lambda(L, t) + \frac{1}{\kappa} \frac{\partial \Lambda(L, t)}{\partial t} + \frac{\Lambda(L, t) (\Lambda(L - h, t)P(L - h, t) - \Lambda(L, t)P(L, t))}{\kappa P(L, t)} + \frac{M(L, t) - M(L - h, t)}{\kappa P(L, T)}, \quad M(L, t) = \int_{0}^{\infty} \lambda^2 p(\lambda, L, t)d\lambda,
\]

which makes it possible to solve the Forward Kolmogorov equation

\[
\frac{\partial p(\lambda, L, t)}{\partial t} = \left( -\kappa \frac{\partial}{\partial \lambda} (\rho(L, t) - \lambda) + \frac{1}{2} \frac{\partial^2}{\partial \lambda^2} \sigma^2 \lambda \right) p(\lambda, L, t) + \lambda (p(\lambda, L - h, t) - p(\lambda, L, t))
\]

in one forward propagation without any need for iterative fitting.
Figure 2: $\rho(L, t)$ calibrated to market skew for iTraxx.
Signature of correlation skew in local intensity

Figure 3: Local intensity consistent with flat Gaussian correlations and with market correlations skew for iTraxx 5y (38) on Oct 12, 2005.
Rules of thumb for intensity and correlation skew

Flat local intensity ⇔ non-stochastic intensity ⇔ no correlations

Linear local intensity ⇔ affine stochastic intensity ⇔ flat correlations, no skew

Non-linear local intensity ⇔ non-affine stochastic intensity ⇔ correlation skew
Calibration of local intensity using an auxiliary static factor model

Calibrate an auxiliary static factor model to the tranches, use the resulting model to obtain $P(L, t)$ everywhere, deduce $\Lambda(L, t)$ by “Dupire” formula

- Gaussian copula mixture
- random factor loadings
- implied (“perfect copula”) of Hull and White
Parametric calibration of local intensity

\[ \bar{L}(t) = E[L_t] - \text{fixed by the underlying credit index} \]

\[ \Lambda(L, t) = \Lambda_0(t) + \sum_{k=1}^{k_{\text{max}}} \alpha_k \left( \frac{L}{\bar{L}(t)} \right)^k, \]

\[ \frac{d\bar{L}(t)}{dt} = \sum_{n=0}^{N} \left( \Lambda_0(t) + \sum_{k=1}^{k_{\text{max}}} \alpha_k \left( \frac{L}{\bar{L}(t)} \right)^k \right) P(L, t). \]

\( \alpha_1, \alpha_2, \ldots \) are fitted to the tranches

\( \Lambda_0(t) - \text{fixed by the decay of the underlying credit index} \)
CDO tranche with option to cancel

Accrual periods $[0, T_1], [T_1, T_2], \ldots [T_{M-1}, T_M]$

Exercise dates $T_{e_1} < T_{e_2} < \cdots < T_{e_s}$

Loss($T_i, k, K$) = $(L_{T_i} - k)^+ - (L_{T_i} - K)^+$

Tranche notional $A_{CDO}(T_i) = (K - k) - \text{Loss}(T_i, k, K)$

Residual protection leg for exercise at $T_e$

$$P_{prot}^{(e)} = \sum_{i>e} (\text{Loss}(T_i, k, K) - \text{Loss}(T_{i-1}, k, K)) D(T_i).$$

Residual fee leg for exercise at $T_e$

$$P_{fee}^{(e)} = \sum_{i>e} S \cdot \tau(T_{i-1}, T_i)(0.5 A_{CDO}(T_{i-1}) + 0.5 A_{CDO}(T_i)) D(T_i).$$

Choice at $T_e$: keep option or exercise into an offsetting residual tranche 
$$\pm (P_{fee}^{(e)} - P_{prot}^{(e)})$$
Backward induction for CDO tranche option

A slice $S_{T_i}$ is a function on discretized state space $(\lambda_i, jh)$ residing at a model date $T_i$.

Conditional expectation on the state achieved at earlier date $T_i' < T_i$ gives a slice at $T_i'$: $S_{T_i'} = E[S_{T_i}|T_i']$, computed using backward Kolmogorov equation.

Option exercise condition is checked recursively,

$$\text{Option}(T_{e_i}) = \max \left( E[\text{Option}(T_{e_{i+1}})|T_{e_i}], \pm E[P^{(e)}_{\text{fee}} - P^{(e)}_{\text{prot}}|T_{e_i}] \right)$$
Numerical results for CDO tranche options: mezzanine

Figure 4: Premium to cancel ATM mezzanine tranches in the 2D Markovian model, $d\lambda_t = \kappa(\rho(L_t, t) - \lambda_t)dt + \sigma \sqrt{\lambda_t}dW_t \ (\kappa = 1.6)$. 

\begin{align*}
\end{align*}
Numerical results for CDO tranche options: equity

Figure 5: Premium to cancel ATM equity tranches in the 2D Markovian model, \( d\lambda_t = \kappa (\rho(L_t, t) - \lambda_t) dt + \sigma \sqrt{\lambda_t} dW_t \) (\( \kappa = 1.6 \)).
Conclusions

• We proposed a 2D intensity-based Markovian model for aggregate loss dynamics with an efficient numerical calibration to CDO tranches. The model can be calibrated to any distribution of loss.

• Associated with every stochastic intensity model is a local intensity model. Local intensity bears the signature of correlation skew in its non-linearity.

• Calibration of local intensity to a sparse set of tranche quotes is not unique. More research is needed to explore the influence of different choices.

• Options on tranches in our model can be priced using backward induction. Option premia can exhibit both increasing and decreasing dependence on the volatility of the intensity.
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