Jump Diffusion in Credit Modeling: a Partial Integro-differential Equation Approach

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November 4, 2006

5th Annual Advances in Econometrics Conference, LSU
Outline

- Structural and first-passage models
- Distance-to-default
- Lévy process: combining Brownian motion and Poisson jumps
- Partial integro-differential equation formulation
- Numerical implementation
- Calibrating single-name default probability term structures
- Modeling credit rating migrations (Markov chain model)
- Concluding remarks
Early Structural Models

Motivation: pricing defaultable bonds (Black and Scholes, 1973 and Merton, 1974)

- $V_t$: total asset of the firm at $t$
- $B_0$: face value of the debt
- $S_t$: equity priced as a call option on $V_t$, struck at $B_0$

\[
S_T = \max(V(T) - B_0, 0)
\]

- $V_T < B_0$ implies a default
- $S_t = S(V_t, B_0)$: based on Black-Scholes option pricing theory
- $B_t = V_t - S_t$: market value of the debt
First-Passage Models

- First introduced by Black and Cox (1976)
- Default can be triggered any time before $T$
- Generalizing the value of the firm $V_t$ to a credit index $X_t$: the distance-to-default
- First passage time:

$$\tau = \inf \{ t \geq 0 : X_t \leq 0 \}$$

- Distribution of $\tau$ may be obtained in some cases
- Defaultable bond price, with recovery $R$:

$$B = e^{-rT} P[\tau > T] + \mathbb{E}[Re^{-r\tau}]P[\tau \leq T]$$
Distance-to-default Models

- Need to calibrate to observed yield spread term structure
- Discrete-time model: Hull and White (2000)
- Continuous-time PDE model: Avellaneda and Zhu (2001)
- Define distance-to-default: \( X_t = V_t - b(t) \),
- For example, \( V_t \) can be the value of the firm, and \( b(t) \) can be the liability
- \( b(t), \ t \geq 0 \) interpreted as a barrier, is to be determined
- Default probability term structure can be matched
Modeling Credit Index with Lévy Process

• Discontinuous process:

\[ dX_t = a(X_t, t)dt + \sigma(X_t, t)dW_t + dq_t, \quad X_0 > 0 \]

• \( W_t \): standard Brownian motion
• \( q_t \) experiences jumps with intensity \( \lambda \)
• \( W_t \) and \( q_t \) assumed to be independent
• Prescribe probability measure of the jump amplitude

\[ G(x, dy) = \mathbb{P}[x \rightarrow (y, y + dy)] \]

• \( u(x, t) \): survival probability density at \( t \)

\[ u(x, t)dx = \mathbb{P}[X_t \in (x, x + dx), \quad t < \tau], \quad x \geq 0 \]
Compound-Poisson Process $q_t$

- Features:
  - Shocks (jumps) coming at random times
  - Markov process
  - $Z_t$: total number of occurrences before $t$,

$$P_n(t) = P\{Z_t = n\} = \frac{(\lambda t)^n}{n!}e^{-\lambda t}$$

- Interpretation of intensity $\lambda$ for small $h$,

$$P_1(h) = \lambda h + o(h), \quad P_0(h) = 1 - \lambda h + o(h)$$

- Applications in finance: stock option pricing (Merton, 1976), reduced-form models (Jarrow and Turnbull, 1992, Duffie and Singleton), structural models (Zhou, 2001)
Sample Paths

Typical Path for Poisson Process

- lambda=10
- lambda=2
Need for Jumps

• Probability of default occurring by $t$:

$$P(t) = P[\tau \leq t] = 1 - Q(t) = 1 - \int_0^\infty u \, dx$$

• With Brownian motion only: default probabilities for short time horizons are extremely low

$$P(t) \to 0, \quad P'(t) \to 0, \quad \text{as} \quad t \to 0$$

• Contradicting market observation

• Partial explanation: risk factors are exogenous

• Several other approaches to correct this: stochastic volatility, constant elasticity of variance, imperfect information models
Difficulty with the Diffusion Model

Default Probability Density with Linear Barriers

\[ b(t) = -\alpha - \beta t \]
Infinitesimal Generator of Lévy Process

- Extending $f = 0$ for $x < 0$
- Allow $\lambda$ to be $x$-dependent
- for $x > 0$

$$A f(x) = \lim_{t \to 0^+} \frac{\mathbb{E}^x [f(X_t)] - f(x)}{t}$$

$$= \frac{1}{2} \sigma^2 f_{xx} + af_x + \lambda(x) \left( \int_0^\infty f(y) G(x, dy) - f(x) \right)$$

- Fokker-Planck forward equation:

$$\frac{\partial u}{\partial t} = A^* u$$
Partial Integro-Differential Equation

• Assumption: \( G(x, dy) = g(x, y)dy = g(y - x)dy \)

• PIDE for the survival density

\[
ut = \frac{1}{2} (\sigma^2 u)_{xx} - (au)_x + \int_0^\infty \lambda(y)u(y)g(x-y)dy - \lambda u, \quad x > 0 \]

• Boundary condition

\[
u(x, t)|_{x=0} = 0, \quad t \geq 0 \]

• Initial condition

\[
u(x, 0) = \delta(x - X_0)\]
Survival Probability Distribution with Jumps

- Brownian motion only
- Brownian motion and Poisson jumps
Jump Amplitude Distributions

- Normal distribution

\[ g(x) = \frac{1}{\sqrt{2\pi\sigma_j^2}} \exp\left(-\frac{(x - \mu_J)^2}{2\sigma_j^2}\right) \]

- Two-sided exponential (Kou, 2002)

\[ g(x) = \begin{cases} 
\frac{\alpha}{\beta_+} e^{-\frac{1}{\beta_+} x} & x > 0, \\
\frac{\alpha}{\beta_+} + \frac{1-\alpha}{\beta_-} & x = 0, \\
\frac{1-\alpha}{\beta_-} e^{\frac{1}{\beta_-} x} & x < 0.
\end{cases} \]

with

\[ 0 < \alpha < 1, \quad \beta_+, \beta_- > 0 \]
Total Mass: the Survival Probability

- Total mass at $t$: $Q(t) = \int_0^\infty u \, dx$
- Monotone decreasing
  \[
  Q'(t) = -\left[\frac{1}{2}(\sigma^2 u)_x\right]_{x=0} + \int_0^\infty \lambda(y)u(y, t)H(y) \, dy \leq 0
  \]
- where
  \[
  H(y) = \int_0^\infty g(x - y) \, dx - 1 \leq 0
  \]
Issues in Numerical Implementation

- Finite difference approximations - implicit-explicit method:
  - Diffusion term treated implicitly for numerical stability
  - Integral term treated explicitly to avoid full matrix
  - Order of the method can be upgraded (Runge-Kutta)
- Smoothing of the initial data ($\delta$ function)
- Smoothing of the kernel, if necessary
- Numerical solutions used in a nonlinear solver/optimization
- Stability is essential to all applications
Default Probability Density with Poisson Jumps

**Exit Probability Density**

- $\lambda_0 = 0$
- $\lambda_0 = 0.5$
- $\lambda_0 = 1$
- $\lambda_0 = 2$

**Default Barrier**

- $\lambda_0 = 0$
- $\lambda_0 = 0.5$
- $\lambda_0 = 1$
- $\lambda_0 = 2$

**Default Intensity**

- $\lambda_0 = 0$
- $\lambda_0 = 0.5$
- $\lambda_0 = 1$
- $\lambda_0 = 2$
• Determine the drift term $a$
• Equivalent to a time-dependent default barrier $b(t) < 0$, and zero drift, requiring
  \[ b'(t) = -a(t) \]
• Initial credit index $X_0 = b(0)$
• Matching condition for $u$ if the default probability $P(t)$ is available:
  \[
  \frac{1}{2} (\sigma^2 u)_x |_{x=0} - \lambda \int_0^\infty \int_0^\infty u(y) g(x - y) \, dy \, dx \\
  = \lambda (P(t) - 1) + P'(t), \quad t > 0
  \]
Example: AAA Banks Default Term Structure

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Default probabilities for the bank industry with AAA ratings, with assumed recovery rate of 50%, July 2005.
Default Barriers for AAA Banks

\[
\begin{align*}
\text{default barriers} \\
\lambda_0 = 0 \\
\lambda_0 = 0.25 \\
\lambda_0 = 0.5 \\
\lambda_0 = 0.75
\end{align*}
\]
Default probabilities for the bank industry with different ratings, with assumed recovery rate of 50%, March 2006.
Default Barriers for Multi-Ratings

![Graph showing default barriers for different ratings over time. The graph plots t vs. b(t) for ratings AAA, AA2, A1, BAA1, BAA3. The x-axis represents time (t), and the y-axis represents b(t). The graph shows the trend of default barriers for each rating over the time period from 0 to 10.]
Credit Yield Spread Perturbation

Yield spreads resulted from different volatility/jump intensity structures.
Application - Credit Rating Migration

- $R(t)$: rating of the entity at $t$
- Assigned by rating agencies, significant price impact
- Historical data available (in transition frequencies)
- Markov chain model (Jarrow, Lando and Turnbull, 1997), transition probability matrix

\[ q_{i,j}(t, T) = P[R(T) = j, |R(t) = i], \quad t < T \]

- Transition across several ratings is possible
- Find the generating matrix $\Lambda$ such that

\[ Q(t) = [q_{i,j}(t)] = e^{t\Lambda} \]
Multi-Barrier Distance-to-Default Model

- Use the distance-to-default \( X_t \) to determine the rating
- Survival region \( x > 0 \) divided into subregions separated by \( b_j, \ j = 1, \ldots, K \)
- Distance-to-default follows the process:

\[
dX_t = a(X_t, t)dt + \sigma(X_t, t)dW_t + dq_t
\]

- Rating is \( j \) at \( t \) if \( b_{j-1} < X_t \leq b_j \)
- Probabilistic approach (Albanese and Chen, 2005)
- Model coefficients can be made space dependent
- In this work, piecewise linear \( a(x), \sigma(x) \) and \( \lambda(x) \) are used
Main Features of the Continuous-Time Model

- Defaults can occur at any time
- Credit upgrades or downgrades represented by barrier crossings
- Continuous time Markov process
- Upgrade and downgrade trend linked to drift term \( \alpha \)
- Correlation for different credit variables can be specified
- Calibration to observable credit rating transition probabilities can be implemented
Forward Equation and Transition Probabilities

- Equation for the survival density $u(x, t)$:
  
  
  \[ u_t + (a(x, t)u)_x = \frac{1}{2} \left( \sigma^2(x, t)u \right)_{xx} + \int_0^\infty \lambda(y)u(y)g(x-y)dy - \lambda u, \quad x > 0 \]

  with boundary condition $u|_{x=0} = 0$

- Transition probability at $T$
  
  \[ q_{j,k} = \int_{b_{k-1}}^{b_k} u^i(x, T)dx \]

  where $u^i$ is a solution to the PIDE with initial condition

  \[
  u^i_0(x) \begin{cases} 
  \geq 0, & b_{j-1} < x < b_j \\
  = 0, & \text{elsewhere}
  \end{cases}
  \]

  satisfying

  \[ \int_{b_{j-1}}^{b_j} u^i_0(x)dx = 1 \]
## Rating Transition Matrix

<table>
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<tr>
<th></th>
<th>AAA</th>
<th>AA</th>
<th>A</th>
<th>BBB</th>
<th>BB</th>
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<td>64.93</td>
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Calibration through Optimization

- **Parameters**
  - drifts $\mathbf{a} = (a_k, k = 1, \ldots, K)$
  - volatilities $\sigma = (\sigma_k, k = 1, \ldots K)$
  - jump intensities $\lambda = (\lambda_k, k = 1, \ldots K)$
  - initial $(u_0^k(x), k = 1, \ldots, n)$ profiles

- **Objective function**
  $$f(\mathbf{a}, \sigma, \lambda) = \| Q_d - Q_{\mathbf{a},\sigma,\lambda} \|^2$$

- $Q_d$: transition probability matrix from data
- $Q_{\mathbf{a},\sigma,\lambda}$: transition probability matrix from the model

- **Optimization package**: L-BFGS-B (version 2.1)
## Fit by the Diffusion Model

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The errors are listed in parentheses, with those larger than 0.5% highlighted.
## Fit by the Jump Diffusion Model

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Fitting the Moody’s Data

Moody’s average rating transition frequency matrix for 1980-2000, normalized for withdrawn ratings.

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<th>Aa</th>
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Parameter structures obtained from optimization, by matching the transition frequency matrices and default probabilities.
Historical default probabilities (Moody’s 1983-1996) and those from the calibrated model.
Change of Measure: Real-World to Risk-Neutral

- Volatility/jump intensity structures provide a base model
- Move from the real world measure to the risk-neutral: change of the drift $a$
- Based on the equivalent Martingale measure (Girsanov’s theorem)
- Equivalent to a default boundary $b(t) \geq 0$
- Not expected to match spreads
- Information about the credit spread as part of the yield spread
- Other consideration should be included, such as taxes
Adjustment in the drift, or the introduction of the modified default boundary, can explain the credit spread in the yield spread of corporate bonds.
Other Applications

- Insight into the generator of the Markov chain
- Provide forward default term structures
- Generate volatility skew
- Unified theory for equity and defaultable bond markets
- Time homogeneity of the transition matrix (for investment grade), vs. cyclic transition matrix (for speculative grade)
Concluding Remarks

- Jumps allowed in distance-to-default
- Additional parameters to fit the data
- In the framework of partial integro-differential equation with variable coefficients
- Efficient and stable numerical algorithms developed
- Short time default probabilities/intensities much more in line with market observation
- Significant improvements in fitting transition probability matrices
- Could explain the credit spread term structure, with other modeling efforts combined
Thank You