Pricing options on film revenue

*This article illustrates two models for cumulative revenues from films, a time-changed gamma process and a compound Poisson process, and how these models can be used to price options. **Don Chance**, **Eric Hillebrand** and **Jimmy Hilliard** find that while both models lead to option prices that accurately reflect discounted future option payouts, the gamma process model is easier to implement*

In recent years, hedge funds and private equity firms have shown an increasing interest in films as a form of investment.[1] It is generally believed that revenues from films are uncorrelated with the investment performance of traditional asset classes, thereby suggesting that films can themselves be considered an asset class. Thus, investors seeking improvements in diversification and possibly new sources of alpha should be interested in direct and indirect investment in films.

Films are, however, an extremely risky form of investment. Highly publicised and expensive films are often dismal failures, while low-budget films, sometimes by independent producers, can achieve surprising success.[2] The financial performance of a film is difficult to predict, indeed even difficult to comprehend, as some films fail in one market and do extremely well in another.[3] As an investment, a film is characterised by a large initial investment and a brief period of concentrated uncertainty. After the first few weeks of distribution, it is usually known if a film will be successful in terms of domestic box-office receipts. As documented in previous studies, however, the majority of films are commercial failures, with a small number of winners providing sufficient returns to offset the large number of losers.[4]

It should not be surprising that the risk associated with entertainment revenue is a prime candidate for specialised financial risk management instruments. One of the first was a $400 million, seven-year Eurobond issued by The Walt Disney Company in 1992. The interest rate was determined by the performance of a combination of 13 Disney films released in Europe. The coupon contained an embedded call option where the underlying is the accumulated revenue. In 1997, pop singer David Bowie issued $55 million of bonds with coupons determined by the revenue from some of his albums. These instruments gave rise to the term Bowie Bonds, but were not successful and were downgraded to junk status in 2004. Bowie Bonds, however, have been offered by at least seven other entertainers.

Risk has reported on a number of attempts to create options based on the performance of entertainment revenues. Conway (1997) describes the creation of an entertainment exchange in London, which was launched with options based on the album Perfect World by pop singer Debbie Bonham. Patel (2004) reports on the efforts of a US company called Center-Group to create an electronic derivatives market on box-office receipts. The Hollywood Stock Exchange (*www.hsx.com*) has operated since 1996 offering virtual stocks and derivatives on film stars and films.[5] The director of new product research at the Chicago Mercantile Exchange even recently mentioned that the exchange is considering offering a futures contract based on film box-office revenue.[6]

The interest in films is not surprising. In a recent report, the Motion Picture Association of America (2007) estimates that US citizens spent almost $9.5 billion on films in 2006. Although US admissions have been on the decline since a peak in 2002, there was an increase in 2006 when 1.45 billion tickets were sold in the US and the average US citizen saw 4.8 films that year. Approximately 600 new films were released in 2006. While the US produces the lion's share of most commercially successful films, the worldwide market is much larger, with a total box-office take of almost $26 billion in 2006. Of course, box-office revenues are only part of the picture as DVD sales and rentals, film-related merchandise, and television showings are major revenue generators.

Financial engineers have shown remarkable creativity in applying their skills to a variety of diverse risk management problems. A derivative instrument to manage the kind of risk associated with a box-office revenue stream offers a potentially valuable tool for this large and highly visible industry and for end-users. In this article, we explore the pricing of options based on film revenues.

[1] *See Kelly (2006) and Marr (2007)*
[2] *Compare* The Alamo, *released in 2004, which included big-name stars Dennis Quaid and Billy Bob Thornton, cost $95 million and brought in $22 million, with* The Blair Witch Project, *made by essentially amateurs in 1999 with a $35,000 budget, which brought in $140 million*
[3] *For example, in 2004 the film* Troy, *costing about $197 million in production and distribution costs, earned only $133 million in US box-office receipts but grossed more than $360 million outside the US, making it one of the top 50 films of all time*
[4] *For studies on the performance and profitability of films, see Ravid & Basuroy (2004), De Vany & Walls (1997, 1999, 2002) and Ravid (1999)*
[5] *That is, no real money is involved in trading. The Hollywood Stock Exchange sells its information to film companies, and thus provides marketing research based on virtual purchases and sales of film stars and films by the general public*
[6] *See Lucchetti & MacDonald (2007)*

This type of instrument is particularly challenging for two reasons. One is that the options are sold before any revenue is accumulated. Hence, the only observation on the underlying at the time of the initial pricing is an accumulated revenue of zero. Another problem is that the underlying is a revenue stream. Hence, it can only increase. Standard diffusion processes for modelling stock prices, commodity prices, interest rates and exchange rates cannot capture the unique characteristics of an accumulating underlying.[7] To address this problem, we propose a model that embeds a deterministic innovation process into a stochastic process with gamma-distributed increments.[8]

A first look at film revenues

To illustrate the characteristics of a box-office revenue stream, we select a sample film, *Along Came A Spider*, which was released by Paramount Pictures on April 6, 2001. The film is rated R in the US, and is a drama starring Morgan Freeman based on a book by popular author James Patterson. The production budget was about \$28 million. It played for 19 weeks in the US and generated box-office revenue of slightly more than \$74 million with another \$31 million outside the US.

Figure 1 illustrates the weekly and cumulative revenue for *Along Came A Spider*. Weekly revenue, shown by the bars, was \$16.7 million the first week and peaked the second week at about \$17.2 million. The third week's take was under \$12 million and all remaining weeks generated less than \$10 million.[9] This pattern of weekly revenue resulted in cumulative revenue, as shown by the line, which is the underlying on which options would be based. Note how cumulative revenue rises steadily and levels off within about two months after release. It is the stochastic process for cumulative revenue that we must model. To do so, we incorporate factors that affect people's willingness to see a film, as well as the marketing efforts of film distributors. We begin with a simple deterministic model as a foundation for a more complex stochastic model.

Modelling options on film revenue

Here, we draw on Chance, Hillebrand & Hilliard (2008) to show how options on film revenues can be priced. A number of studies have looked at modelling the stochastic properties of film revenues.[10] As a foundation for our model, we propose the approach developed by Bass (1969). The hazard rate function for the time to adoption is:

$$h(t) = \frac{f(t)}{1 - F(t)} = p + qF(t) \tag{1}$$

where $f(t)$ is a probability density function at time t, and $F(t)$ is the corresponding cumulative distribution function.

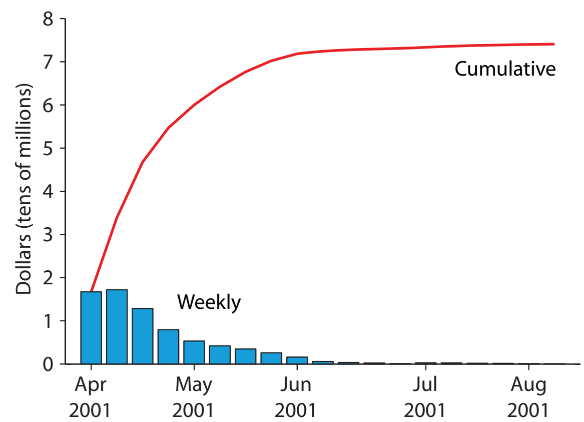
The coefficient $p > 0$ defines how an innovation is adopted independently of the influence of others, while the coefficient $q \geq 0$ determines how existing adoptions affect the probability for more adoptions. With initial condition $F(0) = 0$, the solution to the ordinary differential equation (1) is the distribution function:

$$F(t) = \frac{1 - e^{-(p+q)t}}{1 + \frac{q}{p} e^{-(p+q)t}} \tag{2}$$

Differentiating $F(t)$ gives the probability density for the time t to adoption by an individual:

$$f(t) = \frac{dF(t)}{dt} = \frac{(p+q)^2 pe^{-(p+q)t}}{\left(p + qe^{-(p+q)t}\right)^2}, \quad 0 < t < \infty \tag{3}$$

1 Weekly and cumulative box-office revenues from *Along Came A Spider*, released April 4, 2001



We follow Bass (1969) and assume a fixed population, m, which represents the maximum number of adopters or the market potential. Then $n(t) = mF(t)$ is the expected cumulative number of adopters at t. Accordingly, $F(t) = n(t)/m$ and is the expected percentage of the population that has adopted by time t. The differential equation for $n(t)$ and its solution are, respectively:

$$\begin{aligned} \frac{dn(t)}{dt} &= \left(m - n(t)\right)\left(p + q\frac{n(t)}{m}\right) \\ n(t) &= m\left(\frac{1 - e^{-(p+q)t}}{1 + \frac{q}{p}e^{-(p+q)t}}\right) = mF(t) \end{aligned} \tag{4}$$

Thus, the rate of adoptions is proportional to the non-adopters that remain, with a coefficient of proportionality of $p + qn(t)/\mathrm{m}$.

A stochastic model is then built from the deterministic Bass model by embedding the adoption variable $n(t)$ into a time-changed gamma-process $N(t)$ that calibrates the first moment of the process to the Bass model:

$$\mathbb{E}N(t) = n(t)$$

Note that in this model, the random variable is the number of adopters $N(t)$ at time t, not the time an individual takes to adopt. The first moment $\mathbb{E}N(t) = n(t)$ follows the deterministic function $f(t)$. Consequently, the differential equation (3) is satisfied by the first moment of our model. The increment $N(t) - N(s)$, $t > s$, follows a gamma distribution with mean $m(F(t) - F(s))$ and variance $m(F(t) - F(s))\beta$, where β is a volatility parameter of the gamma process.

For details of the construction of the time-changed gamma-process, see Chance, Hillebrand & Hilliard (2008). The most important feature is that mean and variance depend on the Bass model such that:

$$\begin{aligned} \mathbb{E}\left(N(t) - N(s)\right) &= m\left(F(t) - F(s)\right) \\ Var\left(N(t) - N(s)\right) &= m\beta\left(F(t) - F(s)\right) \end{aligned} \tag{5}$$

Note that the probability density in the Bass model, $dF(t) = f(t)dt$,

[7] *Bakshi & Madan (2002) consider an accumulating underlying in the context of catastrophe loss options*
[8] *Full details of the model and the econometric estimation of parameters are contained in Chance, Hillebrand & Hilliard (2008)*
[9] *Film revenue and other information is taken from* www.the-numbers.com *and* www.imdb.com
[10] *See, for example, De Vany & Walls (2004) and Sawhney & Eliashberg (1996)*

can be interpreted here as an intensity. It determines the average number of adoptions and the variance of adoptions between times s and t. The properties of the model are such that both moments attenuate rapidly under the appropriate choice of parameters. This is an empirical feature of film box-office data. Moreover, the zero initial value of film revenue is not a problem. In the construction of a gamma process, the moments in (5) are proportional to $t - s$. Our transformation makes them proportional to $F(t) - F(s)$. In this sense, we construct a time-changed gamma process.

We write the model as:

$$dN(t) \sim Gamma\left(\frac{mdF(t)}{\beta}, \beta\right) \tag{6}$$

reflecting the statistical property that for arbitrary increments $t - s > 0$:

$$N(t) - N(s) \sim Gamma(\alpha(s,t), \beta) \tag{7}$$

where $\alpha(s, t) = m(F(t) - F(s))/\beta$. Note that the model features a single source of randomness.

Our interest is in valuing a cash-settled European-style call option. We assume that the asset class is uncorrelated with the market portfolio and, consequently, the risk is non-priced.[11] Because the underlying is the number of adoptions, we incorporate the average ticket price in the form of a variable a. The option value at time s for an option expiring at time T is derived under the physical measure as:

$$\begin{aligned} C(s,T) &= ae^{-r(T-s)}\mathbb{E}_s\left[\max\{0, N(s) + N(s,T) - K_N\}\right] \\ &= ae^{-r(T-s)}\int_d^\infty (N(s) + x - K_N) f_{N(s,T)}(x)dx \end{aligned} \tag{8}$$

where r is the risk-free rate, $N(s)$ is adoption through time s, $N(s, T) = N(T) - N(s)$ is the unknown remaining adoption to time T, K_N is the strike price in units of adoptions, $K = aK_N$ is the strike price in dollars, and $d := \max\{0, K_N - N(s)\}$. Plugging in the gamma density $f_{N(s,T)}$, we can write the call option formula as:

$$C(s,T) = ae^{-r(T-s)}\left(\alpha(s,T)\beta\Pi_1 + (N(s) - K_N)\Pi_2\right) \tag{9}$$

where $\alpha(s, T)\beta = m(F(T) - F(s))$:

$$\Pi_1 = 1 - \frac{\gamma(\alpha(s<T)+1, d/\beta)}{\Gamma(\alpha(s,T)+1)}$$

and:

$$\Pi_2 = 1 - \frac{\gamma(\alpha(s,T), d/\beta)}{\Gamma(\alpha(s,T))}$$

is the probability of finishing in-the-money. $\gamma(\alpha, d) = \int_0^d t^{\alpha-1}e^{-t}dt$ is the lower incomplete gamma function. Note that the term Π_2 is somewhat like $N(d_2)$ in the Black-Scholes-Merton model, but that measure is the risk-neutral probability of exercise. Because we price under the physical measure, Π_2 is the actual probability of exercise.

Chance, Hillebrand & Hilliard (2008) show that boundary conditions are met. Some other useful results can be obtained. The forward price of remaining revenue at time s is:

$$J(s,T) := a\mathbb{E}_s N(s,T) = a(n(T) - n(s)) = am(F(T) - F(s))$$

which means that the call value can be expressed in terms of the forward price as:

$$C(s,T) = e^{-r(T-s)}\left(J(s,T)\Pi_1 - (K - R(s))\Pi_2\right) \tag{10}$$

where $R(s) = aN(s)$.

The value of a put can be found following the same approach or by applying put-call parity:

$$P(s,T) = e^{-r(T-s)}\left((K - R(s))(1-\Pi_2) - J(s,T)(1-\Pi_1)\right)$$

The value of the revenue stream is:

$$V(s,T) = e^{-r(T-s)}\left(R(s) + J(s,T)\right)$$

This formula could be used to determine a value at which to securitise the revenue stream.

If $R(s) > K$, the option is irreversibly in-the-money, $d = 0$, and the option value is:

$$C(s,T) = e^{-r(T-s)}\left(J(s,T) + R(s) - K\right)$$

Here the volatility of the underlying has no effect on the option's value.[12]

An alternative model

We propose here a compound Poisson process as an alternative model to the time-changed gamma process. Consider the model:

$$N(t) = \sum_{i=1}^{Z(t)} y_i \tag{11}$$

where $N(t)$ denotes adoption as before, $y_i \sim Gamma(1/\delta, \delta)$ are jumps, and:

$$\mathbb{P}(dZ(t) = 1) = mf(t)dt + o(dt)$$

so that $Z(t)$ is a Poisson count process with time-varying intensity $mf(t)$. This intensity is the same as used above in the construction of the gamma process. Revenue is then given by $R(t) = aN(t)$, where a is the ticket price. This model has the same number of parameters as the gamma-process model. The parameter vector is $\theta = (m, p, q, \delta)$. Note that there are two sources of randomness, the Poisson arrival process and the gamma-distributed jumps. Theorem 1 shows that the time dynamics of the first and second moment can be matched to those of the gamma-process model.

■ **Theorem 1.** The first and second moments of the compound Poisson adoption process N are for any time increment $T - s$, $s \le T$:

$$\mathbb{E}[N(T) - N(s)] = m(F(T) - F(s))$$

$$Var[N(T) - N(s)] = m(1+\delta)(F(T) - F(s))$$

Hence, if we choose $\delta = \beta - 1$, the first two moments of the compound Poisson process coincide with those of the gamma process.

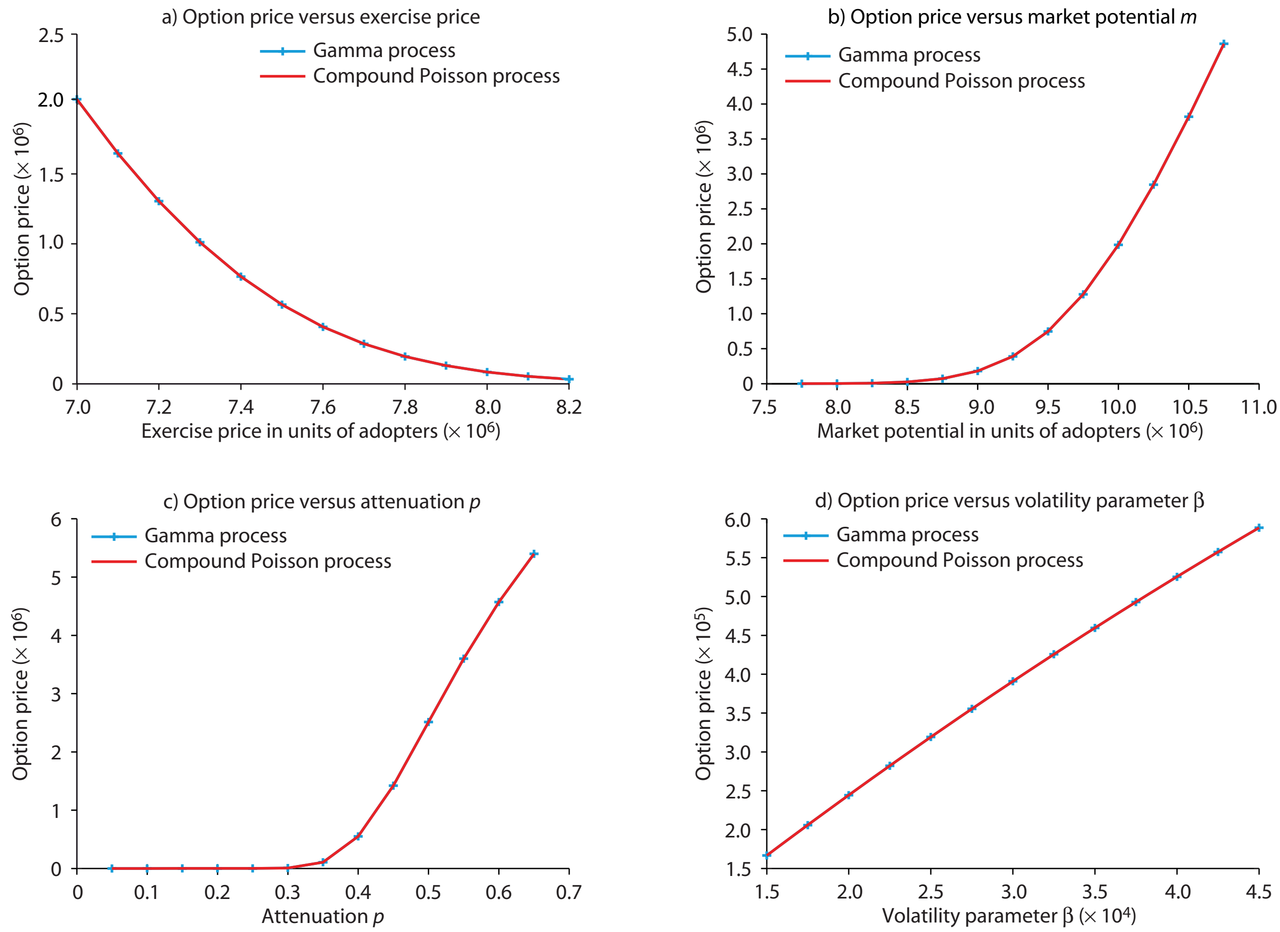
The proof is given in appendix A. Again, we make the assumption that the revenue process is uncorrelated with the market, and hence we price under the physical measure. Theorem 2 presents the call option price; the proof is also found in the appendix.

■ **Theorem 2.** The option price of a European-style call struck at K_N (in units of adoption) with maturity T at time s is given under model (11) as:

[11] *The problem of assuming a zero risk premium for non-traded assets is an issue that certainly applies here as well as to real options and other non-traded assets. In particular, the growing body of research on weather derivatives has not been able to resolve the question of how those instruments can be priced when weather is not a traded asset, nor can it even be held and considered an asset. There is no perfect solution beyond assuming that a comparable traded asset exists, that investors have well-defined utility functions, or that certain combinations of conditions exist, such as in Rubinstein (1976) or Brennan (1979). Alternatively, it may be the case that the asset is uncorrelated with equity returns in practice, thereby suggesting a zero risk premium. Chance, Hillebrand & Hilliard (2008) test this hypothesis by constructing an index of weekly film revenue and find that it has nearly zero correlation with the S&P 500 over the years 1998–2000 using bivariate Granger-causality tests*

[12] *The characteristic that volatility has no effect on option value if an option cannot expire out-of-the-money is not unique to this type of option. The same result can be seen with standard European-style options at certain nodes of a binomial tree in which expiry out-of-the-money cannot occur*

2 Greeks of the option price formulas according to the time-changed gamma process and the compound Poisson process for *Along Came A Spider*



$$e^{r(T-s)}C(s,T) = a\sum_{n=0}^{\infty}\left[\mathbb{P}\left(Z(s,T)=n\right)\left\{n\left(1-\frac{\gamma(n/\delta+1,d/\delta)}{\Gamma(n/\delta+1)}\right) -\left(K_N - N(s)\right)\left(1-\frac{\gamma(n/\delta,d/\delta)}{\Gamma(n/\delta)}\right)\right\}\right] \tag{12}$$

where:

$$\mathbb{P}\left(Z(s,T)=n\right)=\frac{e^{-m(F(T)-F(s))}m^n\left(F(T)-F(s)\right)^n}{n!}$$

Numerically, it is not feasible to evaluate the infinite sum and a truncation number of jumps has to be determined. We find that for our application to film revenue, it usually suffices to evaluate the region between $m(F(T) - F(s)) - 20{,}000$ and $m(F(T) - F(s)) + 20{,}000$ jumps. This certainly depends on the data and needs to be found by trial and error.

Implementing the models

Chance, Hillebrand & Hilliard (2008) show how the gamma-process model can be implemented. Estimates of the parameters m, p, q and β are required. These parameters represent the market potential, the coefficient of innovation or attenuation, the coefficient of imitation or acceleration, and a volatility parameter of the gamma process. Given a time series of film revenue, estimation of the parameter vector $\theta = (m, p, q, \beta)$ is straightforward. The likelihood function is given as:

$$L\left(\theta\middle|\{N(t)\}_{t=1,\ldots,T^*}\right)=\prod_{t=1}^{T^*}\frac{1}{\beta^{\alpha(t-1,t)}\Gamma\left(\alpha(t-1,t)\right)}N(t-1,t)^{\alpha(t-1,t)-1}e^{-N(t-1,t)/\beta} \tag{13}$$

where $\alpha(t-1, t) = m(F(t) - F(t-1))/\beta$ and $N(t-1, t) = N(t) - N(t-1)$, $N(0) = 0$, as before. T^* is the sample size (as opposed to the maturity T of the option). The evaluation of the likelihood function is straightforward; the transformation to the log-likelihood is standard. The log-likelihood function is numerically well behaved.

The likelihood function of the compound Poisson process with parameter vector $\theta = (m, p, q, \delta)$ is given as:

$$L\left(\theta\middle|\{N(t)\}_{t=1,\ldots,T^*}\right)=\prod_{t=1}^{T^*}\sum_{n=0}^{\infty}\mathbb{P}\left(Z(t-1,t)=n\right)f\left(N(t-1,t);n/\delta,\delta\right) \tag{14}$$

where $\mathbb{P}(Z(t-1, t) = n)$ as in theorem 2 and $f(N(t-1, t); n/\delta, \delta)$ is the probability density function of a *gamma*(n/δ, δ)-distributed random variable. There are a couple of numerical issues with this likelihood function. First, the log-likelihood transformation does not decouple the product of the Poisson probability with the gamma density due to the infinite sum. Second, for each observation, the number of jumps considered must be truncated, since the evaluation of the infinite sum is not feasible. The probability from the Poisson source of randomness has $n!$ in the denominator, and the factorial function can be evaluated by most programming environments only up to a small number (for example, 170 in the case of Matlab). This problem can be circumvented by coding:

$$\frac{e^{-m(F(T)-F(s))}m^n\left(F(T)-F(s)\right)^n}{n!}=$$

$$\exp\left(-m\left(F(T)-F(s)\right)+n\log m+n\log\left(F(T)-F(s)\right)-\sum_{j=1}^{n}\log j\right)$$

Still, the truncation number of jumps remains as a nuisance parameter in the likelihood. We choose $\max\{m(F(t)-F(t-1)), N(t-1,t))\}$, since $m(F(t)-F(t-1))$ is the expected number of jumps, and so this will be the maximum of the Poisson probability with respect to n. If the number of jumps is $N(t-1,t)$, each admission is one jump, and this is a plausible maximum for our application. Third, the evaluation of (14) is computationally very slow compared with the evaluation of (13). Finally, convergence of the maximum likelihood estimator is much more difficult to achieve for (14) than for (13). There is a high sensitivity to the choice of initial values for θ in (14). Intuitively, the reason is the double stochasticity in (14). For an arbitrary choice of θ, most often the maximum of $\mathbb{P}(Z(t-1,t)=n)$, which is $m(F(t)-F(t-1))$, will not coincide on the n-axis with the maximum of the gamma density $f(N(t-1,t), n/\delta, \delta)$. Then, the product of the two probability expressions will be numerically indistinguishable from zero. For any given observation, there is a region of plausible values for θ; outside this region, (14) returns meaningless numerical values. In summary, the compound Poisson process is a statistically much more difficult model than the gamma process.

Updating the estimates

In pricing an option on a film revenue stream, we need an estimate of the parameter vector before revenue data is available and an update algorithm as revenue observations arrive. Our proposed approach is to collect box-office data for a large sample of films, estimate the parameters from this sample using maximum likelihood, conduct a cross-sectional regression of the estimated parameters on characteristics that should influence the revenue stream, and use these regression parameters to estimate the parameters for out-of-sample films. We then update these estimates as the film progresses throughout its run using Markov-chain Monte Carlo simulations in a Bayesian update. We use the gamma-process model for this approach since the likelihood of the compound Poisson process is for this purpose prohibitively slow to evaluate.

For our sample, we obtained revenue data on the 100 most successful films in each of 1998, 1999 and 2000. Restricting the sample to only those films that began and ended during that period left us with 244 films. Using data on the average ticket price, available at *www.natoonline.org*, we converted the revenue series to an adoption series. Boswijk & Franses (2005) document that the imitation parameter q cannot be consistently estimated without data beyond the peak of adoption. In addition, we conducted simulations that revealed that the maximum likelihood estimators are of poor quality if $q \neq 0$. Thus we assume $q = 0$ and, therefore, require estimates of only three parameters, m, p and β.

These parameters are estimated for each film in the sample of 244 films. We then conduct cross-sectional regressions of the logs of these parameters on certain variables that have been noted in the literature to be related to film revenue. These variables are the initial number of screens, the log of the budget, and dummy variables for rating categories PG, PG-13 and R with G the omitted class.[13] In these regressions, the R^2 values ranged from 23% to 36%. With respect to rating, the more adult-rated the film is (that is, PG versus G, PG-13 versus PG and R versus PG-13), the lower is β, which is consistent with a previous finding in the literature (DeVany & Walls, 2002) that R-rated films are less risky. PG-13 and R-rated films have higher attenuation p and R-rated films have lower potential m.

[13] *See, for example, Ravid (1999) and Sawhney & Eliashberg (1996)*

Appendix A: proof of theorem 1

Denote $N(s,T) := N(T) - N(s)$ and $Z(s,T) := Z(T) - Z(s)$. The distribution of $Z(s,T)$ is:

$$\mathbb{P}\left(Z(s,T)=n\right)=e^{-m(F(T)-F(s))}\frac{m^n\left(F(T)-F(s)\right)^n}{n!}$$

The moment-generating function of $N(s,T)$ is:

$$\begin{aligned}M_{N(s,T)}(\theta)&=\mathbb{E}e^{\theta N(s,T)}\\&=\mathbb{E}\left[\mathbb{E}\left[e^{\theta\sum_{i=Z(s)+1}^{Z(T)}y_i}\middle|\left\{Z(t)\right\}_{t\le T}\right]\right]\\&=\mathbb{E}\left[\left(\mathbb{E}e^{\theta y_i}\right)^n\right]\text{ for } Z(s,T)=n\in\mathbb{N}\end{aligned}$$

By the assumption that $y_i \sim gamma(1/\delta, \delta)$ for all i, we have:

$$\left(\mathbb{E}e^{\theta y_i}\right)^n=\left(\frac{1}{1-\delta\theta}\right)^{\frac{n}{\delta}}$$

Therefore:

$$\begin{aligned}M_{N(s,T)}(\theta)&=\sum_{n=0}^{\infty}e^{-m(F(T)-F(s))}\frac{m^n\left(F(T)-F(s)\right)^n}{n!}\left(\frac{1}{1-\delta\theta}\right)^{\frac{n}{\delta}}\\&=e^{-m(F(T)-F(s))}e^{m(F(T)-F(s))\left(\frac{1}{1-\delta\theta}\right)^{\frac{1}{\delta}}}\\&=e^{m(F(T)-F(s))\left(\left(\frac{1}{1-\delta\theta}\right)^{\frac{1}{\delta}}-1\right)}\end{aligned}$$

For this function, it is convenient to obtain the moments from the cumulant-generating function:

$$\log M_{N(s,T)}(\theta)=m\left(F(T)-F(s)\right)\left(\left(\frac{1}{1-\delta\theta}\right)^{\frac{1}{\delta}}-1\right)$$

as:

$$\mathbb{E}N(s,T)=\frac{d}{d\theta}\log M_{N(s,T)}(\theta)\bigg|_{\theta=0}=m\left(F(T)-F(s)\right)$$

and:

$$VarN(s,T)=\frac{d^2}{d\theta^2}\log M_{N(s,T)}(\theta)\bigg|_{\theta=0}=m\left(F(T)-F(s)\right)(1+\delta)$$

Appendix B: proof of theorem 2

We begin with the standard expression for the option price as the discounted conditional expected value:

$$
\begin{aligned}
e^{r(T-s)}C(s,T) &= a\mathbb{E}_s\big(N(T)-K_N\big)_+ \\
&= a\mathbb{E}_s\big(N(T)-N(s)+N(s)-K_N\big)_+ \\
&= a\mathbb{E}_s\left(\sum_{i=Z(s)+1}^{Z(T)} y_i + N(s)-K_N\right)_+ \\
&= a\mathbb{E}_s\left[\mathbb{E}\left[\left(\sum_{i=Z(s)+1}^{Z(T)} y_i + N(s)-K_N\right)_+ \middle| \{Z(t)\}_{t\le T}\right]\right] \\
&= a\mathbb{E}_s\left[\mathbb{E}\left[\max\left\{0, \sum_{i=Z(s)+1}^{Z(T)} y_i + N(s)-K_N\right\} \middle| \{Z(t)\}_{t\le T}\right]\right] \\
&= a\mathbb{E}_s\left[\mathbb{E}\left[\mathbf{1}_{\left\{\sum_{i=Z(s)+1}^{Z(T)} y_i > K_N - N(s)\right\}} \right.\right. \\
&\qquad \left.\left.\left(\sum_{i=Z(s)+1}^{Z(T)} y_i + N(s)-K_N\right) \middle| \{Z(t)\}_{t\le T}\right]\right] \\
&= a\mathbb{E}_s\left[\mathbb{P}\left(\sum_{i=Z(s)+1}^{Z(T)} y_i > K_N - N(s) \middle| \{Z(t)\}_{t\le T}\right)\right. \\
&\qquad \times\mathbb{E}\left[\sum_{i=Z(s)+1}^{Z(T)} y_i + N(s)-K_N \middle| \{Z(t)\}_{t\le T}, \right. \\
&\qquad \left.\left.\sum_{i=Z(s)+1}^{Z(T)} y_i > K_N - N(s)\right]\right]
\end{aligned}
$$

By the distribution assumption for the jumps $y_i \sim gamma(1/\delta, \delta)$, we obtain:

$$
\sum_{i=Z(s)+1}^{Z(T)} y_i \sim gamma\left(\frac{Z(s,T)}{\delta}, \delta\right)
$$

Therefore, for $d = \max\{K_N - N(s), 0\}$:

$$
\mathbb{P}\left(\sum_{i=Z(s)+1}^{Z(T)} y_i > d \middle| \{Z(t)\}_{t\le T}\right) = 1 - \frac{\gamma\big(Z(s,T)/\delta, d/\delta\big)}{\Gamma\big(Z(s,T)/\delta\big)}
$$

where:

$$
\gamma\big(Z(s,T)/\delta, d/\delta\big) = \int_0^{d/\delta} u^{Z(s,T)/\delta-1} e^{-u} du
$$

The conditional expected value of the payout is:

$$
\begin{aligned}
&\mathbb{E}\left[\sum_{i=Z(s)+1}^{Z(T)} y_i + N(s)-K_N \middle| \{Z(t)\}_{t\le T}, \sum_{i=Z(s)+1}^{Z(T)} y_i > K_N - N(s)\right] \\
&= \frac{\int_d^\infty xf\big(x; Z(s,T)/\delta, \delta\big)dx}{\mathbb{P}\left(\sum_{i=Z(s)+1}^{Z(T)} y_i > d \middle| \{Z(t)\}_{t\le T}\right)} + N(s) - K_N
\end{aligned}
$$

where:

$$
\int_d^\infty xf\big(x; Z(s,T)/\delta, \delta\big)dx = Z(s,T)\left(1 - \frac{\gamma\big(Z(s,T)/\delta+1, d/\delta\big)}{\Gamma\big(Z(s,T)/\delta+1\big)}\right)
$$

The expression in the theorem for $\mathbb{P}(Z(s, T) = n)$ is immediate from the construction of the Poisson process. We therefore obtain the call option price as:

$$
\begin{aligned}
&e^{r(T-s)}C(s,T) \\
&= a\mathbb{E}_s\left[\int_d^\infty xf\big(x; Z(s,T)/\delta, \delta\big)dx + \right. \\
&\qquad \left.\left(1 - \frac{\gamma\big(Z(s,T)/\delta, d/\delta\big)}{\Gamma\big(Z(s,T)/\delta\big)}\right)\big(N(s)-K_N\big)\right] \\
&= a\sum_{n=0}^\infty \left[\mathbb{P}\big(Z(s,T)=n\big)\left\{n\left(1 - \frac{\gamma(n/\delta+1, d/\delta)}{\Gamma(n/\delta+1)}\right)\right.\right. \\
&\qquad \left.\left. - \big(K_N - N(s)\big)\left(1 - \frac{\gamma(n/\delta, d/\delta)}{\Gamma(n/\delta)}\right)\right\}\right]
\end{aligned}
$$

Alternatively, the proof could depart from the known characteristic function of the compound Poisson process, following the results on option pricing with the fast Fourier transform in Carr & Madan (1999).

Example

We now illustrate the model using data from *Along Came A Spider*, which was released in 2001 and, therefore, is not contained in the sample in which these parameters were estimated. Using the coefficients from the cross-sectional regression, we obtain initial parameter estimates of m = 9,251,245, p = 0.3872 and β = 30,072.[14] We then create a hypothetical four-week call option with an exercise price of $43 million, which is the budget of $28 million plus a $15 million profit.[15] The call option value is found from the gamma process in equation (9) as $391,952 and from the compound Poisson process in equation (12) as $391,951.

The maximum likelihood estimate of the parameter vector (m, p, β) from the gamma-process model is (13137709, 0.3206, 31412) and the maximum likelihood estimate of the parameter vector (m, p, δ) from the compound Poisson model is (17419436, 0.3309, 32143). The most substantial discrepancy is in the estimate of m. After 19 weeks in cinemas, *Along Came A Spider* had collected $74,058,698 in box-office receipts. The market potential implied by the compound Poisson process estimate is 17,419,436 × $5.65 = $98,419,811. The market potential implied by the gamma process estimate is 13,137,709 × $5.65 = $74,228,054.

Figure 2 evaluates the option price formulas (9) from the gamma process (blue) and (12) from the compound Poisson process (red) for a number of different parameter settings. Note that both models imply the same price in all scenarios. Panel (a) illustrates the convex relationship between option price and exercise price, which is characteristic of standard options. Panels (b), (c) and (d) present the Greeks. In the gamma-process model, β is the

[14] *At an average ticket price of $5.65, maximum revenue potential is predicted to be about $52 million*
[15] *There is nothing special about a four-week option and indeed longer periods might be necessary. The decision is entirely up to the distributor selling the option. Also, in practice the exercise price should be scaled to reflect the percentage of revenues expected to be received by the seller of the option*

A. Option values with Bayesian updated and constant parameters through the life of the option

	Option value	
Week	Bayesian update	Constant parameters
0	$391,952	$391,952
1	$159,309	$203,853
2	$7,200,713	$3,884,548
3	$12,150,772	$9,004,864
	Payout	
4	$11,689,149	$11,689,149

volatility measure ($\delta = \beta - 1$ in the compound Poisson process). We see in panel (d) that, in accordance with intuition and standard options, the option value varies positively with volatility. In panel (c) we see a positive relationship between option value and p, the coefficient of innovation. As intuition suggests, the more powerful the effect of how individuals choose to attend a film independent of the influence of others, the more valuable is the option. But this effect is reflected only over a certain range. As we can see, at roughly below 0.40, there is virtually no impact on option value and a declining rate of increase is observed as p increases. In panel (b) we see the effect of market potential, which is clearly a positive factor. In fact, option value increases at an increasing rate with market potential.

Given the rapid resolution of uncertainty as a film is released, more accurate valuation during the option's life is obtained if arriving information is quickly incorporated into the valuation process. Accordingly, Chance, Hillebrand & Hilliard (2008) propose a Bayesian procedure using Markov-chain Monte Carlo for updating estimates of the model parameters. *Along Came A Spider* brought in about $16.7 million the first week, $17.2 million the second week, $12.9 million the third week and $7.9 million the fourth week, resulting in cumulative revenue of about $54.7 million at the expiry of our hypothetical option. Table A illustrates how updating the information set affects the valuation of the option. Column 2 shows the option value that reflects the effects of the arrival of new information and the accumulated revenue to that point. Column 3 shows the option value if we maintain the parameters at their original estimates, although the accumulated revenue to that point is reflected in these values. Naturally, the same value is obtained at the start as well as at expiry. But the valuations during the life of the option are quite different, with the Bayesian updated values being much higher because of the increase in revenue after week one, which can be seen in figure 1.

Conclusion

Films are an increasingly popular asset class for hedge funds and private equity investors. But modelling the process by which film revenue is generated is complex and challenging. Film production and distribution is highly risky and expensive. Box-office revenue is associated with a rapid resolution of uncertainty. These concerns make it such that standard models for valuing assets and derivatives are not easily applicable, if at all. In this article, we contrast the gamma-process model proposed for film revenue in Chance, Hillebrand & Hilliard (2008) with a compound Poisson process. We show that both stochastic processes lead to very similar option values. The numerical difficulties posed by the compound Poisson process, however, are substantial. The gamma-process model is statistically much more tractable. It remains for the industry to bring this type of instrument to the market. We hope that these findings will help stimulate interest. ■

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