An ARCH Model for Several Time Scales

PRELIMINARY AND INCOMPLETE

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Multiple Time Scales in Volatility

- Time Scales = Serial Correlation, Persistence, Mean Reversion Speed
- SV Models with single vol-driver, ARCH, GARCH, PARCH, QARCH, TARCH,...: One time scale
- 2 scales do better: Gallant and Tauchen (2001), Chernov et al. (2003), Fouque et al. (2003)

Multiple Scales SV Models

- $$\begin{aligned} d\log S_t &= \text{drift} + f(X_t, Y_t)dW_t^{(1)}, \\ dX_t &= \vartheta_X(\mathbb{E}X - X_t)dt + \alpha_X dW_t^{(2)}, \\ dY_t &= \vartheta_Y(\mathbb{E}Y - Y_t)dt + \alpha_Y dW_t^{(3)} \end{aligned}$$

- GARCH limit (Nelson 1990, Duan 1997, Corradi 2000)

$$\begin{aligned} d\log S_t &= \mu dt + \sqrt{X_t}dW_t^{(1)}, \\ dX_t &= (\omega - \vartheta X_t)dt \ (+\alpha X_t dW_t^{(2)}), \\ \vartheta &= 1 - \alpha - \beta \end{aligned}$$

- Consequences for option pricing (K. Solna's talk)...

Problem

Is there a way to estimate and separate several time scales in volatility using traditional time series methods?

GARCH and Several Time Scales

- GARCH:

$$r_t := \log S_t - \log S_{t-1} = \mu(t) + \varepsilon_t,$$
$$\varepsilon_t|_{\mathcal{F}_{t-1}} \sim \mathcal{N}(0, h_t)$$
$$h_t = \omega + \sum_{j=1}^{q} \alpha_j \varepsilon_{t-j}^2 + \sum_{j=1}^{p} \beta_j h_{t-j}.$$

  Engle (1982), Bollerslev (1986)

- Phenomenon:

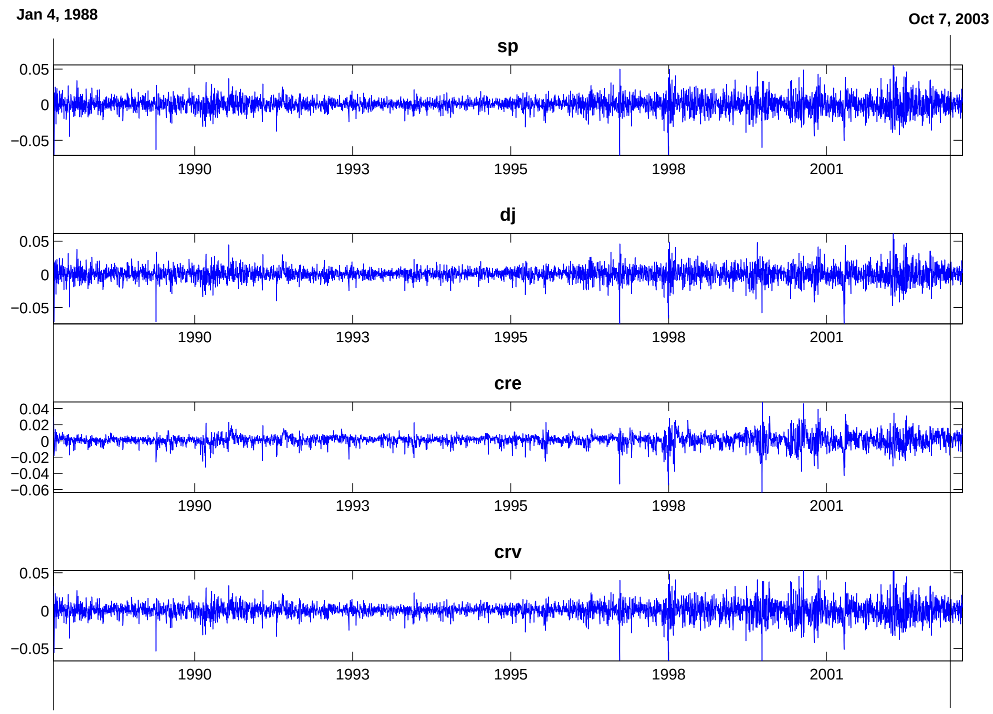
$$\sum_{j=1}^{q} \hat{\alpha}_j + \sum_{j=1}^{p} \hat{\beta}_j \approx 1$$

Conventional Wisdom about Volatility

- High persistence, long scale
- IGARCH, fractional integration ("long memory")
  (Engle and Bollerslev 1986, Ding et al. 1993, Ding and Granger 1996, Baillie et al. 1996, ...)
- Structural breaks and fractional integration
  (Diebold and Inoue 2001, Granger and Hyung 1999, Granger and Teräsvirta 2001)
- Structural breaks and GARCH
  Mikosch and Starica (2005), H. (forthcoming)

Data

Stock price indices (S&P500, DJIA, CRSP equal, CRSP value-weighted) Jan 4, 1988 through Oct 6, 2003. Holding sample Oct 7, 2003 through Dec 31, 2003.



GARCH(1,1)

$$h_t = \omega + \alpha\varepsilon_{t-1}^2 + \beta h_{t-1}$$

	sp	dj	cre	crv
$\hat{\alpha} + \hat{\beta}$	0.995	0.990	0.963	0.991
L	12939	12869	14771	13262
MAE(1)	7.24e-5	4.63e-5	3.36e-5	7.08e-5
MAE(20)	6.24e-5	5.33e-5	3.99e-5	5.76e-5
MAE(60)	4.59e-5	4.06e-5	3.95e-5	4.48e-5

The Short Scale

- If GARCH is correct model, then

$$\nu_t := \varepsilon_t^2 - h_t$$
$$\mathbb{E}\nu_t = 0$$
$$\mathbb{E}\nu_t^2 < \infty, \quad \mathbb{E}\nu_t\nu_{t-j} = 0 \forall j > 0.$$

- Fitting an AR(1) model to $\hat{\nu}_t$:

$$\hat{\nu}_t = \phi\hat{\nu}_{t-1} + \eta_t$$

	sp	dj	cre	crv
$\hat{\phi}$	0.82	0.80	0.79	0.78

- Forecast error improvement over GARCH(1,1)

	sp	dj	cre	crv
MAE(1)	0.10	0.44	0.42	0.19
MAE(20)	0.76	0.81	0.95	0.81
MAE(60)	0.89	0.91	0.98	0.92

Extensions of ARCH to Accommodate Several Scales

- Component GARCH, Engle and Lee (1999)

$$h_t - q_t = \alpha(\varepsilon_{t-1}^2 - q_{t-1}) + \beta(h_{t-1} - q_{t-1}),$$
$$q_t = \omega + \rho q_{t-1} + \phi(\varepsilon_{t-1}^2 - h_{t-1}).$$

- ρ captures long scale, $\alpha + \beta$ captures short scale
- Separation of Scales

	sp	dj	cre	crv
$\hat{\alpha} + \hat{\beta}$	0.94	0.94	0.91	0.93
$\hat{\rho}$	0.9990	0.9995	0.9988	0.9993

- In-Sample and Out-of-Sample Fit

	sp	dj	cre	crv
L	12956***	12889***	14787***	13285***
MAE(1)	1.58	1.80	1.09	1.37
MAE(20)	2.02	2.25	1.34	1.87
MAE(60)	2.78	2.95	1.42	2.40

- Heterogeneous ARCH, Mueller et al. (1997)

$$h_t = \omega + \sum_{j=1}^{n} \alpha_j \left(\sum_{i=1}^{j} \varepsilon_{t-i} \right)^2$$

- Which returns to include?
- Relax: index set

$$K = (k_1,\, k_2,\, \ldots,\, k_m).$$

Then

$$h_t = \omega + \sum_{j \in K} \alpha_j \left(\sum_{i=1}^{j} \varepsilon_{t-i} \right)^2 .$$

- Arbitrary: 1-day, 20-day, 60-day, 100-day, 250-day

- Separation of Scales

	sp	dj	cre	crv
significant returns	(1,20,60)	(1,20,60,250)	(1,20,60)	(1,20,60,100)

- In-Sample and Out-of-Sample Fit

	sp	dj	cre	crv
L	12702(-237)	12707(-162)	14298(-474)	12960(-302)
MAE(1)	0.62	0.62	0.25	0.50
MAE(20)	1.00	1.16	0.88	0.99
MAE(60)	1.17	1.31	0.92	1.09

ARCH and HARCH as QARCH

- Quadratic ARCH, Sentana (1995)

$$h_t = \omega + \epsilon^q(t)^T A \epsilon^q(t),$$

where

$$\epsilon^q(t) = \begin{bmatrix} \varepsilon_{t-1} \\ \varepsilon_{t-2} \\ \vdots \\ \varepsilon_{t-q} \end{bmatrix}$$

$A \in R^{(q-1)\times(q-1)}$ symmetric positive definite

- ARCH(q) is QARCH with

$$A = \begin{bmatrix} \alpha_1 & 0 & \dots & 0 \\ 0 & \alpha_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \alpha_q \end{bmatrix}.$$

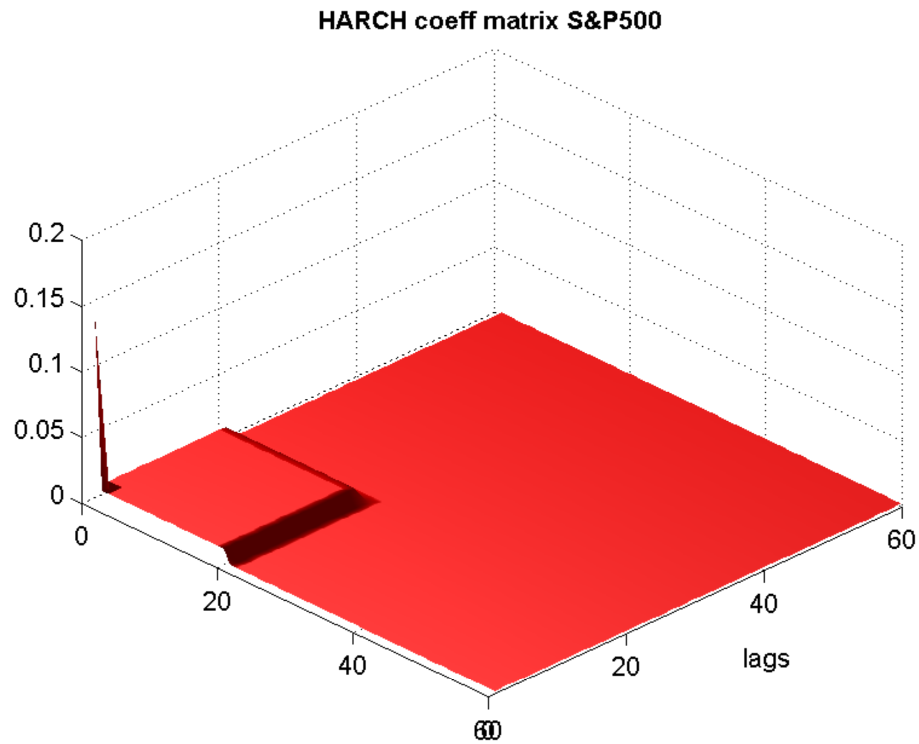
- HARCH is QARCH with

$$A = \begin{bmatrix} \sum_{j\in K} \alpha_j & \sum_{K\ni j>1} \alpha_j & \sum_{K\ni j>2} \alpha_j & \cdots & \sum_{K\ni j>i-1} \alpha_j & \cdots & \alpha_{k_m} \\ \sum_{K\ni j>1} \alpha_j & \sum_{K\ni j>1} \alpha_j & \sum_{K\ni j>2} \alpha_j & \cdots & \sum_{K\ni j>i-1} \alpha_j & \cdots & \alpha_{k_m} \\ \sum_{K\ni j>2} \alpha_j & \sum_{K\ni j>2} \alpha_j & \sum_{K\ni j>2} \alpha_j & \cdots & \sum_{K\ni j>i-1} \alpha_j & \cdots & \alpha_{k_m} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ \sum_{K\ni j>i-1} \alpha_j & \sum_{K\ni j>i-1} \alpha_j & \sum_{K\ni j>i-1} \alpha_j & \cdots & \sum_{K\ni j>i-1} \alpha_j & \cdots & \alpha_{k_m} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ \alpha_{k_m} & \alpha_{k_m} & \alpha_{k_m} & \cdots & \alpha_{k_m} & \cdots & \alpha_{k_m} \end{bmatrix}$$

- General structure of coefficient matrix

$$A = \begin{bmatrix} a_1 & a_2 & a_3 & \cdots & a_{k_m-1} & a_{k_m} \\ a_2 & a_2 & a_3 & \cdots & a_{k_m-1} & a_{k_m} \\ a_3 & a_3 & a_3 & \cdots & a_{k_m-1} & a_{k_m} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{k_m-1} & a_{k_m-1} & a_{k_m-1} & \cdots & a_{k_m-1} & a_{k_m} \\ a_{k_m} & a_{k_m} & a_{k_m} & \cdots & a_{k_m} & a_{k_m} \end{bmatrix} \in \mathbb{R}^{k_m \times k_m}.$$

- For the S&P series:



An Extension of HARCH for Several Scales

- Include coefficients for the displacements:

$$A =$$

$$\begin{bmatrix} a_1 & r_1 a_2 & r_2 a_3 & \cdots & r_{k_m-2} a_{k_m-1} & r_{k_m-1} a_{k_m} \\ r_1 a_2 & a_2 & r_1 a_3 & \cdots & r_{k_m-3} a_{k_m-1} & r_{k_m-2} a_{k_m} \\ r_2 a_3 & r_1 a_3 & a_3 & \cdots & r_{k_m-4} a_{k_m-1} & r_{k_m-3} a_{k_m} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ r_{k_m-2} a_{k_m-1} & r_{k_m-3} a_{k_m-1} & r_{k_m-4} a_{k_m-1} & \cdots & a_{k_m-1} & r_1 a_{k_m} \\ r_{k_m-1} a_{k_m} & r_{k_m-2} a_{k_m} & r_{k_m-3} a_{k_m} & \cdots & r_1 a_{k_m} & a_{k_m} \end{bmatrix}$$

- a_i as given in HARCH, α_j retain interpretation as coeffs of square of j-period returns
- r_j record influence of autocovariance at displacement j

- Specify number of k-day returns to include; specify number of j-day displacements to include. Then search for best specification.
- Selection criteria: In-sample fit, out-of-sample fit.
- Stationarity condition:

$$\sum_{j \in K} j\alpha_j < 1$$

Computational Issues

- Allow for a maximum lag of L.
- Capture K different squared k-day returns as HARCH.
- Capture J different j-day displacements as YARCH.
- $$\binom{L}{K} \times \binom{L}{J}$$
- $L = 250$, $K = 3$, $J = 3$:
$$\binom{L}{K} \times \binom{L}{J} = 6.6e12$$
- Restrict: Use $k = j$, $K = J$ and always include lag one.

- Then,

$$\binom{L}{K-1}$$

- $L = 250$, $K = 3$:

$$\binom{L}{K-1} = 31,125$$

- SuperMike

Toy Application

- $L = 40,\ K = 3$:

$$\binom{L}{K-1} = 780$$

- Separation of Scales

In-Sample Fit: Likelihood

	sp	dj	cre	crv
YARCH	(1,8,39)	(1,8,36)	(1,3,5)	(1,3,4)
L	15232***	15173***	17654***	15681***
HARCH	(1,8,39)	(1,9,38)	(1,3,36)	(1,8,39)
L	15201	15154	17562	15604
GARCH(1,1)-L	12939	12869	14772	13262

Out-of-Sample Fit: MAE(1)

	sp	dj	cre	crv
YARCH	(1,3,11)	(1,12,38)	(1,10,14)	(1,3,11)
MAE(1)	0.50	0.40	0.03	0.23
HARCH	(1,13,33)	(1,3,12)	(1,8,14)	(1,9,23)
MAE(1)	0.55	0.48	0.01	0.40

Out-of-Sample Fit: MAE(20)

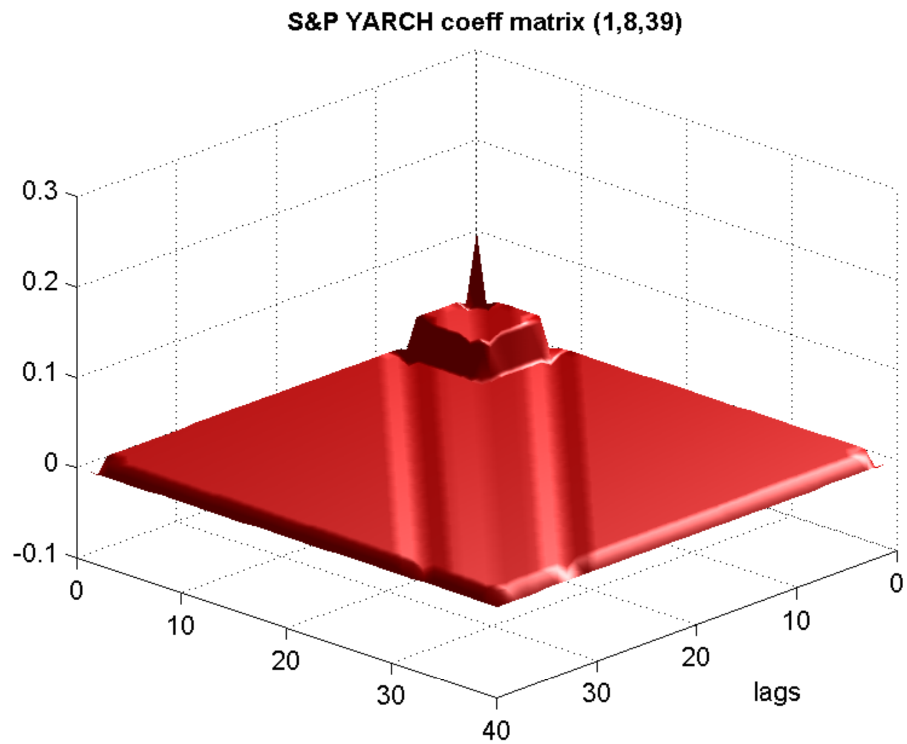
	sp	dj	cre	crv
YARCH	(1,3,15)	(1,12,38)	(1,20,23)	(1,3,15)
MAE(20)	0.82	0.97	0.79	0.79
HARCH	(1,15,28)	(1,30,36)	(1,20,25)	(1,3,26)
MAE(20)	0.91	1.01	0.78	0.87

Out-of-Sample Fit: MAE(60)

	sp	dj	cre	crv
YARCH	(1,3,4)	(1,12,38)	(1,20,23)	(1,3,36)
MAE(60)	1.00	1.11	0.89	0.95
HARCH	(1,13,38)	(1,30,36)	(1,20,25)	(1,36,38)
MAE(60)	1.06	1.14	0.88	1.00

- YARCH: **3** (8), **12** (3), **38** (3)
- HARCH: **3** (3), **8** (3), **36** (4), **38** (3)
- Both: **3** (11), **8** (5), **36** (6), **38** (6)

- For the S&P series and best in-sample fit:



Conclusions / Outlook

- HARCH model (Mueller et al. 1997) lends itself to extensions to capture several scales.
- Approach: Researcher specifies number of time scales, estimation results in location and numerical value of coefficients.
- Allowing autocovariances of excess returns to influence daily conditional volatility increases in-sample and out-of-sample fit.
- Frequently selected lags for daily data in the $\binom{40}{2}$-problem: 3, 8–12, 36–38.

- Extension performs better than HARCH, Component-ARCH, or GARCH(1,1).
- Due to combinatorial complexity of selection problem, high computing power is necessary.
- $\binom{250}{2}$, $\binom{250}{3}$-problems