The Informational Role of Prices and the
Essentiality of Money in the Lagos–Wright Model

L. Araujo, B. Camargo, R. Minetti and D. Puzzello

December 26, 2009

Abstract

A major concern in modern monetary theory has been the development of models where money is essential and yet substantive issues can be analyzed in a tractable manner. At the center of this effort is Lagos and Wright (2005). The tractability in their framework is due to the fact that trade alternates between a centralized market and a decentralized market. A widespread view is that money is essential in the Lagos–Wright framework as long as agents can only observe prices in the centralized market. In this paper we show that this is not the case. Indeed, the presence of centralized trading allows agents to use prices to sustain cooperation. This suggests that an important friction for the essentiality of money is the lack of institutions that allow agents to coordinate behavior.

1 Introduction

Modern monetary theory is based on the notion that one must be explicit about the underlying frictions that render money essential.\footnote{Money is essential if it achieves socially desirable allocations that cannot otherwise be achieved.} Two frictions are considered to be particularly relevant for essentiality: limited commitment and limited record-keeping. In fact, it is commonly agreed that limited commitment and limited record-keeping are necessary for the
essentiality of money.\textsuperscript{2} A less established, but still widespread view, is that as long as one is careful about how the distribution of preferences and technologies in the economy prevents the recurrence of double-coincidences, the absence of both commitment and record-keeping suffice to render money essential. In particular, it is believed that the essentiality of money does not hinge on whether exchange takes place in decentralized or centralized markets. This belief has granted much flexibility in the recent effort towards building models where money is essential and yet substantive issues can be analyzed in a tractable manner. At the center of this effort is Lagos and Wright (2005) (henceforth LW). The main contribution of LW is constructing an environment where, unlike the search models of money in the tradition of Kiyotaki and Wright (1989), money is divisible and the distribution of money holdings is degenerate. The key element of LW is that trade alternates between a centralized market and a decentralized market.\textsuperscript{3}

In this paper we examine whether centralized trading in fact has no implications for the essentiality of money in LW. We are not the first ones to examine this question. Aliprantis, Camera and Puzzello (2007) (henceforth ACP) show that if individual actions are observable in the centralized market, then money can fail to be essential. Lagos and Wright’s response to ACP’s result, see Lagos and Wright (2008), is that in LW agents only observe prices in the centralized market, and thus ACP’s critique does not apply. Unlike ACP, we assume that agents only observe prices in the centralized market and ask whether this indeed guarantees the essentiality of money. Our answer is no.

The environment we consider is the same as in LW but for two features that we now discuss. First, while LW assumes a continuum of agents, we consider large finite populations. In any model, the assumption of a continuum of agents is done for tractability, and is only justifiable if it has no substantive economic implications. To put it differently: “The rationale for using the continuum-of-agents model is that it is a useful idealization of a situation with a large finite number of agents, but if equilibria in the continuum model are radically different from equilibria in the model with a finite number of agents, then this idealization makes little


\textsuperscript{3}An alternative to LW is Shi (1997), who considers an environment without centralized markets.
sense" (Levine and Pesendorfer (1995), p. 1160). Thus, we want to ensure that the argument in Lagos and Wright (2008) does not hinge on the continuum population assumption.

Second, while in LW the centralized market is Walrasian, we model the centralized market as a non-cooperative game. More precisely, we model it as a market game along the lines of Shapley and Shubik (1977). This way we maintain the centralized market as a large anonymous market where all agents observe the same price and prices are the only conduits of information. The main reason for taking the non-cooperative approach is that the assumption of price-taking behavior is understood to be an idealization of behavior in large markets where individual agents have little market power. Market games have been used in the literature to provide non-cooperative foundations for Walrasian markets, see Mas-Colell (1982) for a survey of this area of research. Another reason for the non-cooperative approach is that if one wants to assess the conditions under which money is essential, one must consider whether agents have the incentive to follow alternative credit-like arrangements. The standard competitive model does not specify how payoffs are defined off the equilibrium path, and thus is ill-suited to check the feasibility of competing mechanisms of exchange.

The main result of the paper is that if agents are patient enough, then there exists a non-monetary equilibrium that implements the first-best. This result holds for any number of agents, and the lower bound on the discount factor does not depend on the population size. The intuition for our result is simple. When the number of agents is finite, individual actions in the centralized market have an impact on prices regardless of the population size. Thus, prices can convey information about individual deviations and this can be used to sustain cooperative behavior. Our main result stands in sharp contrast to Araujo (2004), who shows that in the Kiyotaki–Wright environment, see Kiyotaki and Wright (1993), autarky is the only non-monetary equilibrium outcome when the population is large enough no matter how patient the agents are. This fundamental difference between the Kiyotaki–Wright framework and the Lagos–Wright framework suggests that the absence of mechanisms that allow agents to coordinate behavior is an important friction for the essentiality of money.

The paper is organized as follows. We introduce our framework in the next section and describe it as an infinitely repeated game in Section 3. We establish our main result in Section 4. The equilibrium construction in Section 4 relies on prices reflecting individual
behavior in a very precise way. This suggests that the inessentiality of money is not robust to the introduction of noise in the price formation process. In Section 5 we show, contrary to this intuition, that the introduction of noisy prices does not restore essentiality. We conclude the paper in Section 6 with a discussion of our results.

2 The Environment

Time is discrete and indexed by $t \geq 1$. There are two stages at each date and preferences are additively separable across dates and stages. The population consists of an even number $N$ of infinitely lived agents with a common discount factor $\delta \in (0, 1)$. We index agents by $j \in \{1, \ldots, N\}$. The two stages differ in terms of the matching process, preferences, and technology. In the first stage, agents are randomly and anonymously matched in pairs. In the second stage, trade takes place in a centralized market. Agents don’t discount payoffs between stages in the same period. It makes no difference for our results if centralized trading takes place first in each period.

In the decentralized market agents trade a divisible special good. There are no double-coincidences and the probability $\alpha$ that an agent is a producer in a match is equal to the probability that he is a consumer. For simplicity, we assume that $\alpha = 1/2$. We obtain the same results if $\alpha < 1/2$. An agent who consumes $q \geq 0$ units of the special good enjoys utility $u(q)$, whereas an agent who produces $q$ units of this good incurs a disutility $c(q)$. The functions $u$ and $c$ satisfy the usual assumptions, i.e., $u(0) = c(0) = 0$, and $u$ and $c$ are strictly increasing and continuously differentiable with $u$ strictly concave and $c$ convex (or $u$ concave and $c$ strictly convex). Moreover, there exists $\bar{q} > 0$ such that $u(\bar{q}) = c(\bar{q})$. Let $q^* > 0$ be the unique solution to $u'(q) = c'(q)$. Surplus is maximized in a match when the producer transfers $q^*$ units of the special good to the consumer.

In the centralized market agents can consume and produce a divisible general good. The market operates as a trading post, which we describe in the next section. Production is as follows. Every unit of effort creates one unit of the general good. There exists an upper bound $\bar{x} > 0$ on the amount of effort an agent can exert in a period. An agent who consumes $x \geq 0$ units of the general good obtains utility $U(x)$, while an agent who produces $x \geq x \geq 0$
units of this good incurs a disutility $x$. The function $U$ satisfies the same assumptions as $u$. If $U$ is strictly concave, let $x^* > 0$ be the unique solution to $U''(x^*) = 1$. When $U(x) = x$, the maximizer of $U(x) - x$ is indeterminate. Both the special good and the general good are perishable across stages and dates.

# 3 The Lagos–Wright Framework as a Repeated Game

In this section we cast our environment as an infinitely repeated game. This is useful when we discuss the robustness of our inessentiality result to the introduction of noise in the price formation process. We start with a description of the stage game.

## 3.1 The Stage Game

The stage game is an extensive form game that consists of one round of trade in the decentralized market followed by one round of trade in the centralized market.

Trade in the decentralized market takes place as follows. First, in every match the agents simultaneously and independently choose from \{yes, no\} after learning whether they are consumers or producers. If either agent in a match says no, then the match is dissolved with no trade taking place. If both agents in a match say yes, i.e., if both agents agree to trade, then the producer transfers $q^*$ units of the special good to the consumer before the match is dissolved.\(^4\)

In the centralized market an agent can produce both for his own consumption and for exchange in a trading post. The sequence of events is as follows. In every $t \geq 1$ each agent $j$ simultaneously and independently chooses: (i) the quantity $z^t_j$ of the general good he produces for his own consumption; (ii) the quantity $y^t_j$ of the general good he produces to the trading post; (iii) the bid $b^t_j \leq y^t_j$ he submits to the trading post. By definition, the price of the general good in period $t$ is

\[
p_t = \frac{\sum_{j=1}^{N} b^t_j}{\sum_{j=1}^{N} y^t_j},
\]

\(^4\)We obtain the same results if the producer can choose the quantity $q$ he transfers to the consumer. The approach we follow simplifies the description of strategies considerably.
where \( p_t = 0 \) if \( \sum_{j=1}^{N} y_j^t = 0 \). The quantity of the general good that \( j \) obtains in the trading post in period \( t \) is then \( x_j^t = b_j^t / p_t \), where we adopt the convention that \( 0/0 = 0 \). Note that \( \sum_{j=1}^{N} y_j^t = \sum_{j=1}^{N} b_j^t / p_t = \sum_{j=1}^{N} x_j^t \), so that the aggregate supply in the trading post is always equal to the aggregate demand. Prices are publicly observable.

In the stage game, autarky corresponds to the situation in which no production takes place in the decentralized market and agents only produce for themselves in the centralized market. When \( U \) is strictly concave, the highest payoff an agent can obtain in autarky is \( U(x^*) - x^* \). When \( U \) is linear, i.e., when \( U(x) = x \), the highest payoff in autarky is zero. We then have the following result.

**Lemma 1.** The following holds in every Nash equilibrium. There exists no production in the decentralized market. In the centralized market: (i) each agent produces and consumes \( x^* \) when \( U \) is strictly concave; (ii) each agent consumes the same amount that he produces when \( U \) is linear.

**Proof:** Consider the case where \( U \) is strictly concave. The proof when \( U \) is linear is very similar. First notice that every agent can guarantee himself a payoff of at least \( U(x^*) - x^* \) in the centralized market. Indeed, an agent can always produce \( x^* \) for his own consumption. Since the highest aggregate payoff that can be achieved in the centralized market is \( N[U(x^*) - x^*] \), we can then conclude that in every Nash equilibrium all agents obtain \( U(x^*) - x^* \) in the centralized market regardless of how they behave in the decentralized market. Hence, the producers in the decentralized market have no incentive to say yes, and so no production takes place in this market in any Nash equilibrium of the stage game.

### 3.2 The Infinitely Repeated Game

Our environment consists of infinite repetitions of the stage game of Subsection 3.1. The history of an agent in every period is the list of: (i) his past actions in both markets; (ii) the actions of his past partners in the decentralized market; (iii) the past prices in the centralized market. A behavior strategy for an agent is a map from the set of all his possible histories into a (mixed) action.

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5This convention is consistent with the fact that if \( p_t = 0 \), then all agents must bid zero in period \( t \), in which case they cannot buy the general good in the market.
We describe strategies by means of automata.\textsuperscript{6} For this, let $A_1 = \{\text{yes, no}\}$ be the action set of an agent in the decentralized market and $A_2 = \{a_2 = (z, y, b) : z, y \in \mathbb{R}_+ \text{ and } b \leq y\}$ be the action set of an agent in the centralized market. In our setting, an automaton is a list $(W, w^0, (f_1, f_2), (\tau_1, \tau_2))$ where: (i) $W$ is a set of states; (ii) $w^0 \in W$ is the initial state; (iii) $f_1 : W \to \Delta(A_1)$ and $f_2 : W \to \Delta(A_2)$ are decision rules in the decentralized and centralized markets, respectively; (iv) $\tau_1 : W \times A_1^2 \to W$ and $\tau_2 : W \times A_2 \times \mathbb{R}_+ \to W$ are transition rules in the decentralized and centralized markets, respectively. A decision rule is a specification of behavior as a function of states. Thus, an agent in state $w$ chooses $f_1(w)$ if he is in the decentralized market and $f_2(w)$ if he is in the centralized market. A transition rule in the decentralized market associates the next state of the automaton with the agent’s current state and the profile of actions in his match. More precisely, $\tau_1(w, a_1, a_1')$ is the new state of an agent who enters the decentralized market in state $w$ if the consumer and producer in his match choose $a_1$ and $a_1'$, respectively. Similarly, a transition rule in the centralized market associates the next state of the automaton with the agent’s current state, his action, and the observed price. We restrict attention to strategy profiles $\sigma$ where all agents behave according to the same automaton.

A profile of states for a strategy profile $\sigma$ is a map $\pi : W \to \{1, \ldots, N - 1\}$ such that $\pi(w)$ is the number of other agents in the population who are in state $w$. Notice that $\sum_{w \in W} \pi(w) = N - 1$. Denote the set of all state profiles by $\Pi$. A belief for an agent is a map $p : \Pi \to [0, 1]$ such that $\sum_{\pi \in \Pi} p(\pi) = 1$, where $p(\pi)$ is the probability the agent assigns to the event that the profile of states is $\pi$. Let $\Delta$ be the set of all possible beliefs. A belief system for an agent is a map $\mu : W \to \Delta$. In an abuse of notation, we use $\mu$ to denote the profile of belief systems where all agents have the same belief system $\mu$.

We consider sequential equilibria of the repeated game. Note that autarky in every period is an equilibrium outcome. The first-best is achieved when in every period trade takes place in all meetings in the decentralized market and all agents consume and produce $x^*$ in the centralized market. In the next section we show that there exists $\delta \in (0, 1)$ independent of the population size such the first-best is also an equilibrium outcome as long as $\delta \geq \tilde{\delta}$.

\textsuperscript{6}See Mailath and Samuelson (2006), Section 2.3, for a discussion of the use of automata to describe behavior in repeated games.
4 The Inessentiality of Money

In this section we construct an equilibrium that sustains the first–best if agents are patient enough. In what follows, $C$ stands for cooperation, $D$ stands for deviation, and $A$ stands for autarky.

Let $\sigma^*$ be the strategy profile where all agents behave according to the following automaton. The set of states is $W = \{C, D, A\}$ and the initial state is $C$. The decision rules $f_1$ and $f_2$ are given by

\[ f_1(C) = f_1(D) = \text{yes}, \quad f_1(A) = \text{no} \]
\[ f_2(C) = (0, x^*, x^*), \quad f_2(D) = (x^*, x^*, 0), \quad \text{and} \quad f_2(A) = (x^*, 0, 0). \]

For instance, an agent in state $C$ agrees to trade in the decentralized market and chooses $(0, x^*, x^*)$ in the centralized market. The transition rules $\tau_1$ and $\tau_2$ are such that

\[ \tau_1(C, a_1, a'_1) = \begin{cases} C & \text{if} \quad (a_1, a'_1) = (\text{yes, yes}) \\ D & \text{if} \quad (a_1, a'_1) \neq (\text{yes, yes}) \end{cases}, \quad \tau_1(D, a_1, a'_1) \equiv D, \quad \tau_1(A, a_1, a'_1) \equiv A, \]
\[ \tau_2(w, a_2, p) = \begin{cases} C & \text{if} \quad w \in \{C, D\} \text{ and } p \in \{1, \frac{N-2}{N}\} \\ A & \text{if} \quad w \in \{C, D\} \text{ and } p \notin \{1, \frac{N-2}{N}\} \end{cases}, \quad \text{and} \quad \tau_2(A, a_2, p) \equiv A. \]

For instance, an agent in state $C$ in the decentralized market remains in $C$ only if there is trade in his match, otherwise he moves to state $D$. Likewise, an agent in state $C$ in the centralized market stays in $C$ if the price he observes is either 1 or $(N - 2)/N$, otherwise he moves to state $A$. Note that under $\sigma^*$ trade takes place in every meeting in the decentralized market and agents always choose $(0, x^*, x^*)$ in the centralized market. Thus, $\sigma^*$ implements the first–best.

Now consider the belief system $\mu^*$ where: (i) an agent in state $C$ believes that all other agents are in state $C$; (ii) an agent in state $A$ believes that all other agents are in state $A$; (iii) an agent in state $D$ believes that there is one other agent in state $D$ and the remaining agents are in state $C$. It is easy to see that $(\sigma^*, \mu^*)$ is a consistent assessment.

In the remainder of this section we assume that agents behave according to $\sigma^*$ and form beliefs according to $\mu^*$. Let $V_{DM}^*$ and $V_{CM}^*$ be the (discounted) lifetime payoffs to an agent.
in state $C$ before he enters the decentralized and centralized markets, respectively. Then,

$$ V_{DM}^* = \frac{1}{1-\delta} \left\{ \frac{1}{2} [u(q^*) - c(q^*)] + U(x^*) - x^* \right\} \quad \text{and} \quad V_{CM}^* = U(x^*) - x^* + \delta V_{DM}^*. $$

Now let $V_A^*$ be the lifetime payoff to an agent in state $A$. Since there is no discounting between stages in the same period,

$$ V_A^* = (1-\delta)^{-1} [U(x^*) - x^*]. $$

Finally, let $V_D^*$ be the lifetime payoff to an agent in state $D$ before he enters the centralized market. In this case, the agent’s action in the centralized market is $(x^*, x^*, 0)$. Since he believes that one other agent chooses the same action and the remaining agents choose $(0, x^*, x^*)$, he expects the price in the centralized market to be $(N - 2)/N$. Thus,

$$ V_D^* = U(x^*) - 2x^* + \delta V_{DM}^*. $$

Notice no agent is ever in state $D$ before he enters the decentralized market (on and off the path of play).

**Proposition 1.** Suppose that $x^* \geq c(q^*)$. There exists $\delta \in (0, 1)$ independent of $N$ such that $(\sigma^*, \mu^*)$ is a sequential equilibrium for all $\delta \leq \delta$.  

The following result is useful in the proof of Proposition 1. It also helps understanding the specification of behavior in the centralized market for the agents in state $D$. Suppose that $N - 2$ agents choose $(0, x^*, x^*)$ in the centralized market and the remaining two agents choose the same action $(Z, Y, B)$. If $(Z, Y, B)$ is such that the price is $(N - 2)/N$, then the highest flow payoff for the two latter agents is $U(x^*) - 2x^*$. Indeed, if $(Z, Y, B)$ is such that

$$ p = \frac{2B + (N - 2)x^*}{2Y + (N - 2)x^*} = \frac{N - 2}{N}, $$

then $BN/(N - 2) = Y - x^*$. This implies that

$$ U(B/p + Z) - (Y + Z) = U(Y + Z - x^*) - (Y + Z), $$

which is maximized when $Y + Z = 2x^*$.  

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Proof: We first check incentives in state $C$. A producer in the decentralized market has no profitable one-shot deviation if

$$-c(q^* + V^*_{CM}) \geq U(x^*) - 2x^* + \delta V^*_{DM},$$

which is satisfied since $x^* \geq c(q^*)$. It is immediate to see from this that a consumer also has no profitable one-shot deviation. Consider now an agent in the centralized market and suppose he chooses $(Z, Y, B)$. In this case, his flow payoff is $U(B/p + Z) - (Z + Y)$, where

$$p = \frac{(N - 1)x^* + B}{(N - 1)x^* + Y}.$$ 

It is easy to see that $B/p$ is increasing in $B$. Since $B \leq Y$, the highest flow payoff the agent can obtain is $U(Y + Z) - (Y + Z)$, which is not greater than $U(x^*) - x^*$. Since $V^*_{DM}$ is the highest continuation payoff possible for the agent, we can then conclude that he has no profitable one-shot deviation.

We now check incentives in state $D$. As already observed, no agent is ever in state $D$ in the decentralized market. Consider then an agent in the centralized market. There are two types of one-shot deviations to consider: (1) the agent behaves so that $p \in \{1, (N - 2)/N\}$, which implies that his continuation payoff is $V^*_{DM}$; (2) the agent behaves so that $p \notin \{1, (N - 2)/N\}$, which implies that his continuation payoff is $V^*_{A}$.

Consider case (1). First notice that since $B \leq Y$, the agent can never behave in a way that makes the price equal to 1. Suppose then that the agent’s action is $(Z, Y, B)$ such that

$$p = \frac{(N - 2)x^* + B}{(N - 1)x^* + Y} = \frac{N - 2}{N}.$$ (2)

This implies that $BN/(N - 2) = Y - x^*$. Hence, the payoff from a one-shot deviation is

$$U(B/p + Z) - (Y + Z) + \delta V^*_{DM} = U(Y + Z - x^*) - (Y + Z) + \delta V^*_{DM},$$

which is maximized when $Y + Z = 2x^*$. Thus, there exists no profitable one-shot deviation.

Consider now case (2). Here, a one-shot deviation is not profitable if

$$U(x^*) - 2x^* + \delta V^*_{DM} \geq U(B/p + Z) - (Y + Z) + \delta V^*_{A},$$

which is equivalent to

$$U(x^*) - 2x^* + \frac{\delta}{2(1 - \delta)} [u(q^*) - c(q^*)] \geq U(B/p + Z) - (Y + Z).$$ (3)
Since $B/p$ is increasing in $B$ and $B \leq Y$, we have that

$$U(B/p + Z) - (Y + Z) \leq U\left(\frac{Y((N-1)x^* + Y)}{(N-2)x^* + Y} + Z\right) - (Y + Z)$$

$$= U\left(Y + Z + \frac{Yx^*}{(N-2)x^* + Y}\right) - (Y + Z) < U(Y + Z + x^*) - (Y + Z).$$

Now observe that the right–hand side of the last inequality is bounded above by $U(x^*)$ and that the left–hand side of (3) increases to infinity as $\delta \uparrow 1$. We can then conclude that there exists $\hat{\delta}$ such that a one–shot deviation is not profitable if $\delta \geq \hat{\delta}$. Notice that $\hat{\delta}$ is independent of the population size.

Finally, it is immediate to see that no one–shot deviation is profitable in state $A$. □

One point that we should emphasize about our strategy of proof is that it is quite different from the strategy of ACP. Their environment is very much like a repeated prisoner’s dilemma in the sense that communicating a defection to the population in the centralized market involves taking an action that is myopically optimal. On the other hand, in our setting, communicating a defection is costly for the agent in terms of flow payoffs. What sustains the threat of punishment is that if an agent deviates off the path of play, then this leads to an even greater punishment (permanent autarky).

To finish, notice that Proposition 1 is true even without the restriction that $x^* \geq c(q^*)$. In our candidate equilibrium we assume that the only punishment to an agent who defects in the decentralized market is his payoff loss in the subsequent round of trading in the centralized market. This happens because cooperation is restored after agents observe a price of $(N-2)/N$. The condition $x^* \geq c(q^*)$ can be dropped if a defection in the decentralized market were to lead to a greater punishment, as it would be the case if a price of $(N-2)/N$ led to a number of periods of autarky.

5 Noisy Prices

The equilibrium construction in the proof of Proposition 1 relies on the fact that prices in the centralized market are sensitive to changes in individual behavior. This begs the question of whether Proposition 1 is true when prices are noisy. When prices in the centralized market
are a random function of individual behavior, our framework becomes a repeated game with noisy observations. Results from Sabourian (1990) then suggest that when the population is large enough, the only possible equilibrium outcomes are the ones where in each period all agents play a Nash equilibrium of the stage game.\footnote{Sabourian (1990) extends results established in Green (1980) in the context of a repeated Cournot game. Al-Najjar and Smorodinsky (2001) derive results that complement the ones in Sabourian (1990).} By Lemma 1, this translates into autarky being essentially the only non–monetary outcome when the number of agents is large. It turns out that this is not true in our case. In what follows we show that the first–best remains an equilibrium outcome irrespective of the population size even when prices are noisy.

The environment is the same as before except that now in every period $t$ each unit of effort directed to production for exchange in the trading post yields $\theta_t$ units of the general good, where the $\theta_t$ are i.i.d. across time with $E[\theta_t] = 1$. Thus, the price $p_t$ in the centralized market in period $t$ is

$$p_t = \frac{\sum_{j=1}^{N} b^j_t}{\theta_t \sum_{j=1}^{N} y^j_t}.$$  

Note that $\sum_{j=1}^{N} b^j_t / p_t = \theta_t \sum_{j=1}^{N} y^j_t$, and so aggregate demand in the trading post is always equal to aggregate supply. The assumption that shocks to production are perfectly correlated across agents is done for simplicity. We assume that the $\theta_t$ are continuously distributed in some interval $[\underline{\theta}, \overline{\theta}]$ with $0 < \underline{\theta} < 1 < \overline{\theta} < \infty$. We obtain the same results if the $\theta_t$ have a discrete distribution.

Given that production for own consumption is not subject to shocks, agent $j$'s period–$t$ consumption in the centralized market if his action is $(z^j_t, y^j_t, b^j_t)$ is

$$E \left[ U \left( \frac{b^j_t}{p_t} + z^j_t \right) \right] = E \left[ U \left( \theta_t \frac{b^j_t \sum_{j=1}^{N} y^j_t}{\sum_{j=1}^{N} b^j_t} + z^j_t \right) \right].$$

Thus, even though $E[\theta_t] \equiv 1$,

$$E \left[ U \left( \theta_t \frac{b^j_t \sum_{j=1}^{N} y^j_t}{\sum_{j=1}^{N} b^j_t} + z^j_t \right) \right] \neq U \left( \frac{b^j_t \sum_{j=1}^{N} y^j_t}{\sum_{j=1}^{N} b^j_t} + z^j_t \right)$$

when agents are risk–averse. In other words, when agents are risk–averse, the same profile of actions in the centralized market leads to different payoffs depending on whether prices are noisy or not. Since the purpose of our robustness exercise is to check whether the
introduction of noise affects the informational role of prices, the natural assumption to make is that \( U(x) = x \), i.e., agents are risk–neutral. In this way, we guarantee that risk considerations do not play a role in the agents’ decisions.

Let \( \sigma^{**} \) be the strategy profile where all agents behave according to the following automaton. As before, the set of states is \( W = \{C, D, A\} \) and the initial state is \( C \). The decision rules \( f_1 \) and \( f_2 \) are given by

\[
\begin{align*}
  f_1(C) &= f_1(D) = \text{yes}, \quad f_1(A) = \text{no} \\
  f_2(C) &= f(A) = (0, 0, 0), \quad \text{and} \quad f_2(D) = (0, 1, 1).
\end{align*}
\]

The transition rules \( \tau_1 \) and \( \tau_2 \) are such that

\[
\tau_1(C, a_1, a'_1) = \begin{cases} 
  C & \text{if } (a_1, a'_1) = (\text{yes, yes}) \\
  D & \text{if } (a_1, a'_1) \neq (\text{yes, yes})
\end{cases}, \quad \tau_1(D, a_1, a'_1) \equiv D, \quad \tau_1(A, a_1, a'_1) \equiv A,
\]

\[
\tau_2(w, a_2, p) = \begin{cases} 
  C & \text{if } w \in \{C, D\} \text{ and } p \in \{0\} \cup [p_0, 1/\theta], \\
  A & \text{if } w \in \{C, D\} \text{ and } p \notin \{0\} \cup [p_0, 1/\theta],
\end{cases}
\]

where \( p_0 \in (1/\theta, 1/\theta) \).

Now let \( \mu^{**} \) be the belief system used in the proof of Proposition 1. Thus, an agent in either \( C \) or \( A \) believes that all other agents are in the same state, while an agent in \( D \) believes that one other agent is in \( D \) and everyone else is in \( C \). It is easy to see that \((\sigma^{**}, \mu^{**})\) is consistent. When have the following result.

**Proposition 2.** There exists \( \delta \in (0, 1) \) independent of the population size such that \((\sigma^{**}, \mu^{**})\) is an equilibrium if \( \delta \geq \delta \).

**Proof:** Let \( V_{DM}^{**} \) and \( V_{CM}^{**} \) be the lifetime payoffs to an agent in state \( C \) before he enters the decentralized market and the centralized market, respectively. Then,

\[
V_{DM}^{**} = \frac{1}{2(1-\delta)}[u(q^*) - c(q^*)] \quad \text{and} \quad V_{CM}^{**} = \delta V_{DM}^{**}.
\]

Now let \( V_{D}^{**} \) be the lifetime payoff to an agent in state \( D \) before he enters the centralized market and \( F \) be the c.d.f. of \( 1/\theta_t \). Since such an agent believes that \( p_t = 1/\theta_t \) and the lifetime payoff to agent in state \( A \) is zero, we then have that

\[
V_{D}^{**} = \delta [1 - F(p_0)] V_{DM}^{**} = [1 - F(p_0)] V_{CM}^{**}.
\]
Note that $F(p_0)$, the probability the economy moves to autarky, is positive since $p_0 > 1/\theta$. As in the equilibrium of Section 4, an agent is never in state $D$ in the decentralized market.

We first check incentives in state $C$. A producer in the decentralized market has no profitable one-shot deviation if

$$-c(q^*) + V^{**}_{CM} \geq [1 - F(p_0)]V^{**}_{CM} \iff \frac{\delta F(p_0)}{2(1 - \delta)} [u(q^*) - c(q^*)] \geq c(q^*),$$

which is satisfied as long as $\delta$ is close enough to one. It is immediate to see that a consumer also has no profitable one-shot deviation (now regardless of $\delta$). Consider now an agent in the centralized market and suppose he chooses $(Z, Y, B)$ with $Y > 0$. Since (he believes) he is the only one to use the trading post, his expected flow payoff is $E[(\theta_t Y + Z) - (Z + Y)]$, which is zero. Moreover, $V^{**}_{DM}$ is the highest continuation payoff possible for the agent. Hence, he has no profitable one-shot deviation.

We now check incentives in state $D$. As already observed, no agent is ever in state $D$ in the decentralized market. Consider then an agent in the centralized market and suppose his action is $(Z, Y, B)$. Suppose first that $Y > 0$. In this case, his expected flow payoff is

$$E\left[\theta_t \frac{B(Y + 1)}{B + 1} - Y\right],$$

which is maximized when $B = Y$, and so is always non-positive. Moreover, his continuation payoff is

$$\left[1 - F\left(p_0 \frac{1 + Y}{1 + B}\right)\right] V^{**}_{CM},$$

which is also maximized when $B = Y$. Recall that $F$ is the c.d.f. of $1/\theta_t$. Suppose now that $Y = 0$. In this case, the agent’s expected flow payoff is zero and his continuation payoff is $[1 - F(p_0)]V^{**}_{CM}$. Indeed, when the agent does not use the trading post, the distribution of prices is the same he obtains if he chooses $f_2(D)$. Consequently, there are no profitable one-shot deviations in $D$.

Finally, it is immediate to see that no one-shot deviation is profitable in state $A$.

6 Concluding Remarks

We consider an environment where exchange in the centralized market is mediated by a Shapley–Shubik trading post. One could ask whether our results survive if we model trade
in the centralized market in a different way. We believe that as long as agents have some market power, no matter how small, our inessentiality result goes through regardless of the details of the price formation mechanism.\footnote{One alternative price formation mechanism would be a double auction. Large double auctions have also been used to provide non-cooperative foundations for competitive markets. See Rustichini et al. (1994) and Cripps and Swinkels (2006).}

Since prices are public, agents observe prices in the centralized market even if they do not use the trading post to exchange goods. Given the role of prices as a coordinating device, it is natural to ask whether our results survive when an agent can observe prices only if he supplies goods to the trading post. It turns out that the assumption of public prices does not play any role in our results. This is clear in Proposition 1. Indeed, the strategy used in its proof has all agents in the economy using the trading post in the centralized market in every period. The same cannot be said about Proposition 2, though, as the strategy used in its proof is such that no agent uses the trading post on the path of play. In the Appendix we construct an equilibrium that sustains the first-best with the property that agents always make use of the trading post on the path of play.

The reason why the first-best is an equilibrium outcome even when the price formation process is noisy is that agents can coordinate their behavior in the centralized market in a way that prices can be used to convey information about deviations in the decentralized market. With noisy prices, we achieve this by keeping the volume of trade in the centralized market small irrespective of the population size. Thus, differently from Sabourian (1990), the actions of a single agent can still have a non-negligible impact on prices even when the population is large.

What emerges from the analysis in this paper is that money can fail to be essential even when there exists no commitment and no record-keeping as long as there are institutions that help agents coordinate their behavior.\footnote{The point that money can fail to be essential in the presence of coordination mechanisms also shows up in Araujo and Camargo (2009), who study the essentiality of money in a random matching economy in which, as in Cavalcanti and Wallace (1999), there exists a record-keeping technology keeps track of the actions of a subset of the population.} In the Lagos–Wright framework, this institution is the centralized market.
References


7 Appendix

Consider the strategy profile $\sigma^{**}$ where all agents behave according to the following automaton. Once more, the set of states is $W = \{A, C, D\}$ and the initial state is $C$. The decision rules $f_1$ and $f_2$ are given by

\[
\begin{align*}
f_1(C) &= f_1(D) = \text{yes}, \quad f_1(A) = \text{no} \\
f_2(C) &= (0, \varepsilon, \varepsilon), \quad f_2(D) = (0, \overline{x}, 0), \quad \text{and} \quad f_2(A) = (0, 0, 0),
\end{align*}
\]

where $\varepsilon > 0$ is such that

\[
\xi = \frac{(N-2)\varepsilon}{\overline{\theta}(N-2)\varepsilon + 2\overline{x}} < 1/\overline{\theta}.
\]

In order to define the transition rules $\tau_1$ and $\tau_2$, let $S = [\xi/\overline{\theta}, \xi/\overline{\theta}] \cup [1/\overline{\theta}, 1/\overline{\theta}]$. Notice that $\xi/\overline{\theta} < 1/\overline{\theta}$ by construction. The transition rules $\tau_1$ and $\tau_2$ are such that

\[
\begin{align*}
\tau_1(C, a_1, a'_1) &= \begin{cases} C \text{ if } (a_1, a'_1) = (\text{yes, yes}) \\
D \text{ if } (a_1, a'_1) \neq (\text{yes, yes}) \end{cases}, \\
\tau_1(D, a_1, a'_1) &\equiv D, \quad \tau_1(A, a_1, a'_1) \equiv A, \\
\tau_2(w, a_2, p) &= \begin{cases} C \text{ if } w \in \{C, D\} \text{ and } p \in S \\
A \text{ if } w \in \{C, D\} \text{ and } p \notin S \end{cases}, \quad \text{and } \tau_2(A, a_2, p) \equiv A.
\end{align*}
\]

Notice that under $\sigma^{**}$ all agents always use the trading post on the path of play. Indeed, $p_t = 1/\theta_t$ if in period $t$ all agents are in state $C$ in the centralized market, in which case $\tau_2$ implies that all agents remain in $C$ with probability one.

Now let $\mu^{**}$ be the belief system of Propositions 1 and 2: an agent in state $w \in \{A, C\}$ believes that all other agents are in the same state and an agent in state $D$ believes that one other agent is in state $D$ and the remaining agents are in state $C$. It is easy to see that $(\sigma^{**}, \mu^{**})$ is consistent.

For the argument that follows, let $\Omega$ and $\omega$ denote the c.d.f. and p.d.f. of $\theta_t$, respectively, and assume that there exists $\lambda > 0$ such that $\omega(\theta) \geq \lambda$ for all $\theta \in [\overline{\theta}, \overline{\theta}]$. This assumption is satisfied, for instance, if $\theta_t$ is uniformly distributed. Moreover, assume that $\overline{x} \geq c(q^*)$. We discuss how both assumptions can be relaxed at the end of the Appendix.

**Proposition 3.** There exists $\delta \in (0, 1)$ independent of the population size such that if $\delta \geq \delta$, then $(\sigma^{**}, \mu^{**})$ is an equilibrium.
Proof: Let $V_{DM}^{***}$ and $V_{CM}^{***}$ be the lifetime payoffs to an agent in state $C$ before he enters the decentralized market and the centralized market, respectively. Then,

$$V_{DM}^{***} = \frac{1}{2(1-\delta)}[u(q^*) - c(q^*)] \quad \text{and} \quad V_{CM}^{***} = \delta V_{DM}^{***}.$$ 

Now let $V_{D}^{***}$ be the lifetime payoff to an agent in state $D$ before he enters the centralized market. Since such an agent believes that $p_t = \xi/\theta_t$, we then have that

$$V_{D}^{***} = -x + \delta V_{DM}^{***} = -x + V_{CM}^{***}.$$ 

As in the equilibria of Sections 4 and 5, an agent is never in state $D$ when he is in the decentralized market. Finally, let $V_{A}^{***} = 0$ be the lifetime payoff to an agent in state $A$.

We first check incentives in state $C$. A producer in the decentralized market has no profitable one-shot deviation if

$$-c(q^*) + V_{CM}^{***} \geq V_{D}^{***} = -x + V_{CM}^{***},$$

which is satisfied by assumption. It is immediate to see that a consumer also has no profitable one-shot deviation (also regardless of $\delta$). An argument similar to the one used in the proof of Proposition 2 shows that an agent in state $C$ the centralized market has no profitable one-shot deviation as well.

We now check incentives in state $D$. There are two types of one-shot deviations we need to consider, one that changes the support of the price distribution and one that does not change this support.

Suppose first that an agent in state $D$ in the centralized market chooses $(Z, Y, B)$ with the property that

$$\bar{\xi} = \frac{B + (N-2)\varepsilon}{Y + x + (N-2)\varepsilon} = \xi.$$ 

This ensures that the agent’s continuation payoff is $V_{DM}^{***}$. Simple algebra shows that $\xi = \bar{\xi}$ implies that $B/\xi - Y + x = 0$, and so the agent’s flow payoff gain from the choice under consideration is zero. Thus, he has no profitable one-shot deviation.

Suppose now the agent chooses $(Z, Y, B)$ such that $\bar{\xi} \neq \xi$. Since

$$\bar{\xi} - \xi \propto 2Bx + B(N-2)\varepsilon + x(N-2)\varepsilon - Y(N-2)\varepsilon$$

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and \( \bar{\pi} \) is the highest amount of effort an agent can exert in the centralized market in a given period (so \( Y \leq \bar{\pi} \)), only one-shot deviations with \( \tilde{\xi} > \xi \) are possible (conditional on \( \tilde{\xi} \neq \xi \)). Let then \( \tilde{\xi} = \alpha \xi \), with \( \alpha > 1 \). Notice that the upper bound for \( \alpha \) is 
\[
\alpha = \frac{\bar{\pi} + (N-2)\varepsilon}{(N-2)\varepsilon}.
\]
and that \( \alpha \xi < 1 \). Hence, \([\alpha \xi/\bar{\theta}, \alpha \xi/\bar{\theta}] \) has a non-empty intersection with \((\xi/\bar{\theta}, 1/\bar{\theta})\) for all \( \alpha \in (1, \bar{\alpha}] \). Thus, every one-shot deviation with \( \tilde{\xi} > \xi \) has a positive chance of triggering permanent autarky. For simplicity, assume that \( \alpha \xi/\bar{\theta} < 1/\bar{\theta} \). This implies that \( \tilde{\xi}/\theta_t \in (\xi/\bar{\theta}, 1/\bar{\theta}) \) if, and only if, \( \theta_t \in (\bar{\theta}, \alpha \bar{\theta}) \). The other case can be dealt with in a similar way. Now observe that \( \tilde{\xi} = \alpha \xi \) implies that
\[
\frac{B}{\xi} - Y + \bar{\pi} = \frac{\alpha - 1}{\alpha} [2\bar{\pi}(N-2)\varepsilon + (N-2)\varepsilon^2] = \frac{\kappa(\alpha - 1)}{\alpha}.
\]
Thus, the gain from a one-shot deviation is
\[
\frac{\kappa(\alpha - 1)}{\alpha} - [V_{DM}^{**} - V_A^{**}] \int_{2}^{\alpha \theta} \omega(\theta) d\theta \leq (\alpha - 1) \left[ \frac{\kappa}{\alpha} - V_{DM}^{**} \theta \lambda \right] < (\alpha - 1)[\kappa - V_{DM}^{**} \theta \lambda].
\]
Since \( V_{DM}^{**} \uparrow \infty \) as \( \delta \uparrow 1 \), we can then conclude that there exists \( \hat{\delta} \in (0, 1) \) such that the agent has no profitable one-shot deviation if \( \delta \geq \hat{\delta} \).

Finally, it is immediate to see that no one-shot deviation is profitable in state \( A \). \( \square \)

As in the proof of Proposition 1, the assumption that \( \bar{\pi} \geq c(q^*) \) can be dropped if we change \( \sigma^{**} \) so that a price in the range \([\xi/\bar{\theta}, \xi/\bar{\theta}]\) leads to a number of periods in autarky. We can accommodate the case where \( \inf\{\omega(\theta) : \theta \in [\bar{\theta}, \bar{\theta}]\} = 0 \) as follows. Since \( \omega \) is continuous and its image contains positive numbers, there exists \( \lambda > 0 \) such that \( \{\theta : \omega(\theta) > \lambda\} \) is a non-empty open set. So, there exists an open interval \((\theta', \bar{\theta}')\) such that \( \omega(\theta) \geq \lambda \) for all \([\theta', \bar{\theta}']\). Take the set \( S \) in the definition of \( \tau_2 \) in \( \sigma^{**} \) to be such that \( S = [\xi/\bar{\theta}, \xi/\bar{\theta}'] \cup [1/\bar{\theta}, 1/\bar{\theta}] \), i.e., now a price in the interval \((\xi/\theta', \xi/\bar{\theta}]\) triggers permanent autarky.