Short communication

Critical agents in networks

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Abstract

We examine the fundamental tension between efficiency and stability in social network formation. Jackson and Wolinsky (JET, 1996) showed that the component-wise egalitarian payoff rule supports an efficient network as being pairwise stable if and only if the network benefit function is critical link monotone. We extend this insight to strong pairwise stability and derive that the critical link monotonicity condition has to be strengthened to a condition on individuals occupying critical positions in the network.

Keywords: Critical link; Critical position; Pairwise stability

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1. Introduction

A fundamental insight arising from the seminal paper of Jackson and Wolinsky (1996) is that there is a tension between stability and efficiency in network formation. A number of subsequent papers have discussed this tension in various network formation scenarios. (See the excellent

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We emphasize here that if it.\(^1\) We analyze a framework introduced by Jackson and Wolinsky (1996) wherein a network produces a certain value and the value is divided among players according to some allocation rule. Attention is restricted to the component-wise egalitarian allocation rule. Jackson and Wolinsky (1996) conclude that the conflict between efficiency and pairwise stability disappears under the component wise egalitarian allocation rule if the network value function satisfies critical link monotonicity, i.e., if both players involved in the critical link have no incentive to delete it. In this note we extend this insight for “strong pairwise stability” in which every pair of players form links under mutual consent and every individual player is allowed to delete multiple links simultaneously.

When critical agents are present, we identify a condition called disintegration proofness under which efficient networks – if they exist – are necessarily strongly pair-wise stable. This condition imposes that these critical agents have no incentives to disconnect the network. Complete absence of critical agents in a network implies that efficient networks are strongly pairwise stable as well.

2. Preliminaries

In this section we define the formal elements used in describing networks.

2.1. Networks and network components

Let \(N = \{1, 2, \ldots, n\}\) be a finite set of players. An (undirected) link between \(i\) and \(j\) is formally defined as the set \(\{i, j\}\). We use the shorthand notation \(ij\) to denote this link. The collection of all potential links on \(N\) is denoted by \(g_N = \{ij \mid i, j \in N \text{ and } i \neq j\}\). An undirected network \(g\) is defined as any collection of links \(g \subseteq g_N\) and the collection of all networks on \(N\) is denoted by \(G^N\). The network \(g_N\) composed of all possible links is called the complete network and the network \(g_0 = \emptyset\) is the empty network.

Let \(\pi : N \rightarrow N\) be a permutation on \(N\). For every network \(g \in G^N\), \(g^\pi = \{\pi(i)\pi(j) \mid ij \in g\} \in G^N\) is the corresponding permuted network. Two networks \(g, h \in G^N\) have the same topology if there is a permutation \(\pi\) on \(N\) such that \(h = g^\pi\).

For every network \(g \in G^N\), and every player \(i \in N\), we denote \(i\)'s neighborhood in \(g\) by \(N_i(g) = \{j \in N \mid j \neq i\ \text{ and } ij \in g\}\). An alternative description is player \(i\)'s link set, \(L_i(g) = \{ij \in g \mid j \in N_i(g)\}\). We also define \(N(g) = \bigcup_{i \in N} N_i(g)\) and let \(n(g) = |N(g)|\) with the convention that if \(N(g) = \emptyset\), we let \(n(g) = 1\).\(^2\)

A path from player \(i\) to player \(j\) in network \(g\) is defined as a set of players \(\{i_1, \ldots, i_m\} \subseteq N\) such that \(i_1 = i\), \(i_m = j\) and \(i_k i_{k+1} \in g\) for every \(k \in \{1, \ldots, m-1\}\). Two players are connected if a path exists between them. The network \(g' \subseteq g\) is a non-empty component of \(g\) if \(n(g') \geq 2\) and for all \(i, j \in N(g')\), \(i \neq j\), there exists a path in \(g'\) connecting \(i\) and \(j\) and for any \(i \in N(g')\) and \(j \in N(g)\), \(ij \in g\) implies \(ij \in g'\). We denote the set of non-empty components of the network \(g\) by \(C(g)\). The set of

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\(^1\)In graph theory, critical players are denoted as cut nodes. A critical link is also known as a bridge.

\(^2\)We emphasize here that if \(N(g) \neq \emptyset\), we have that \(n(g) \geq 2\). Namely, in those cases the network has to consist of at least one link.
isolated players in $g$ given by $N_0(g)=N\setminus N(g)=\{i\in N\mid N_i(g)=\emptyset\}$. We follow the Jackson and Wolinsky (1996) convention of treating an isolated player as an empty component:

$$C(g) = \{h\mid h \in C(g)\} \cup \{\{i\} \mid i \in N_0(g)\}. \quad (1)$$

Furthermore, we define a node based partitioning of the network $g$

$$\Gamma(g) = \{N(h)\mid h \in C(g)\} \cup \{\{i\} \mid i \in N_0(g)\} \quad (2)$$
as the partition of the player set $N$ based on the component structure $C(g)$ of $g$.

### 2.2. Value functions and allocation rules

Following Jackson and Wolinsky (1996), we describe the “value” generated by participation in a network by means of a network value function given by $\nu: G^N \rightarrow \mathbb{R}$ assigning a totally generated utility value $\nu(g) \in \mathbb{R}$ to the network $g \in G^N$ such that $\nu(g_0)=0$. The space of all network value functions $\nu$ such that $\nu(g_0)=0$ is denoted by $V^N$. The sub-space of all non-negative network payoff functions is denoted by $V^N_+$.

A network value function $\nu \in V^N$ is component additive if $\nu(g) = \Sigma_{h \in C(g)} \nu(h)$. An immediate consequence of component additivity is the fact that isolated players $i \in N_0(g)$ generate no value.

A network $g \in G^N$ is efficient with respect to value function $\nu$ if $\nu(g) \geq \nu(g')$ for all $g' \subseteq g_N$.

The payoff to an individual player is determined by an allocation rule $Y: G^N \times V^N \rightarrow \mathbb{R}^N$. Here, $Y_i(g, \nu)$ is the payoff to player $i$ from the network $g$ under the value function $\nu$. We now define some appealing properties for an allocation rule.

Let $\pi: N \cong N$ be a permutation. Now $\nu^\pi$ is defined by $\nu^\pi(g^\pi) = \nu(g)$.

(i) An allocation rule $Y$ is balanced if $\Sigma_{i \in N} Y_i(g, \nu) = \nu(g)$ for all $\nu$ and $g$.

(ii) An allocation rule $Y$ is component balanced if $\Sigma_{i \in N(h)} Y_i(g, \nu) = \nu(h)$ for every $g$ and $h \in C(g)$ and every component additive $\nu$.

Note that component balance along with component additivity implies that fully disconnected players in $N_0(g)$ always have an allocated payoff of zero.

Let $\nu \in V^N$. The component-wise egalitarian allocation rule is defined by

$$Y_i^{ce}(g, \nu) = \frac{\nu(h_i)}{n(h_i)} \quad (3)$$

where $h_i \in C(g)$ such that $i \in N(h_i)$ and $h_i=\emptyset$ if there is no $h \in C(g)$ such that $i \in N(h)$.

The component-wise egalitarian allocation rule $Y^{ce}$ is the unique allocation rule that is component balanced and assigns an equal payoff to all players in the same component of a network. Also, we note that $Y^{ce}(\cdot, \nu)$ is balanced for every component additive $\nu \in V^N$, but not for arbitrary network value functions.

### 2.3. Stability concepts

We now describe three fundamental link-based network stability properties. Denote by $g+ij$ the network obtained by adding link $ij$ to the existing network $g$, i.e., $g+ij=g \cup \{ij\}$. Similarly, $g-ij$ denotes the network that results from deleting link $ij$ from the existing network $g$, i.e., $g-ij=g \setminus \{ij\}$. 

Let $Y$ be some allocation rule. Then we define:

(i) A network $g \in G^N$ is link deletion proof ($LDP$) if for every player $i \in N$ and every neighbor $j \in N_i(g)$, it holds that $Y(g - ij, v) \leq Y(g, v)$.

(ii) A network $g \in G^N$ is strong link deletion proof ($SLDP$) if for every player $i \in N$ and every set of neighbors $M \subseteq N_i(g)$, it holds that $Y_i(g \setminus h_M, v) \leq Y_i(g, v)$ where $h_M = \{ij \in g | j \in M\} \subseteq L_i(g)$.

(iii) A network $g \in G^N$ is link addition proof if for all players $i, j \in N$, it holds that $Y(g + ij, v) \geq Y(g, v)$ implies $Y(g + ij, v) < Y(g, v)$.

Jackson and Wolinsky (1996) introduced link deletion proofness and link addition proofness, although they did not explicitly define these concepts as such. Strong link deletion proofness was introduced recently by multiple authors. For a thorough discussion of these equilibrium concepts we refer to Bloch and Jackson (2005).

These three fundamental stability concepts can be used to define additional stability concepts. A network $g \in G^N$ is pairwise stable if $g$ is link deletion proof and link addition proof. Furthermore, a network $g \in G^N$ is strongly pairwise stable if it is strong link deletion proof and link addition proof. The only difference between these two stability concepts is that under strong pairwise stability individual players are allowed to remove multiple links rather than a single link under their control. Strong pairwise stability is a natural stability concept in a situation where links are formed only with bilateral consent of both players involved in link formation but links can be deleted unilaterally.

We conclude our discussion with the introduction of the notion of critical link monotonicity introduced by Jackson and Wolinsky (1996).

**Definition 2.1.** A link $ij \in g \in G^N$ is critical in the network $g$ if $\# g - (g - ij) < \# (g - ij)$. The pair $(g, v)$ satisfies critical link monotonicity if for any critical link $ij \in h$ with $h \in C(g)$ and the two associated components $h_1, h_2 \in \Gamma(g - ij)$, we have that

$$v(h) \geq v(h_1) + v(h_2) \implies \frac{v(h)}{n(h)} \geq \max \left[\frac{v(h_1)}{n(h_1)}, \frac{v(h_2)}{n(h_2)}\right]$$

(Jackson and Wolinsky, 1996) showed that critical link monotonicity constitutes a necessary and sufficient condition for the existence of efficient networks that are pairwise stable with regard to the component wise egalitarian allocation rule:

**Claim 1.** If $g$ is efficient relative to a component additive $v$, then $g$ is pairwise stable for $Y^{ce}$ relative to $v$ if and only if $(g, v)$ satisfies critical link monotonicity. (Jackson and Wolinsky, 1996, Claim, page 61)

However, critical link monotonicity is not adequate for a similar result for strong pairwise stability. We demonstrate this next using a simple example.

**Example 2.2.** Let $N = \{1, 2, 3, 4, 5\}$. Let the network value function $v$ be given by $v(\{ij, ih\}) = 9$, $v(\{ij, ih, ik, jm\}) = 10$, and $v(g) = 0$ for all other $g \in G^N$. Hence, there are only two network topologies generating a positive value in $v$: A three player chain network and networks with a topology depicted in Fig. 1. Observe that $v$ is component additive and anonymous.

Now consider the component-wise egalitarian allocation rule $Y^{ce}$ for this particular setup. Clearly, every efficient network is of the topology depicted in Fig. 1. However, such a network is
not strongly pairwise stable because it is not SLDP. Indeed, $Y_i^{ce}(\{ij,jh\}) = \frac{9}{3} = 3$ and $Y_i^{ce}(\{ij,ih,ik,jm\}) = \frac{10}{5} = 2$. Therefore, player $i$ would sever two of his three links: $Y_i^{ce}(\{ij,ih,ik,jm\} \setminus \{ih,ik\}) = Y_i^{ce}(\{ij,jm\}) = 3 > 2 = Y_i^{ce}(\{ij,ih,ik,jm\})$.

In the network $g = \{ij,ih,ik,jm\}$ all links are critical. Consider deletion of any link and let $g_1$ and $g_2$ where possibly $g_2 = \emptyset$ be the two associated components.\(^3\) Thus, $v(g) = 10$, $v(g_1) = 0$ and $v(g_2) = 0$, implying that $\frac{v(g)}{n(g)} = 2$, $\frac{v(g_1)}{n(g_1)} = 0$, and $\frac{v(g_2)}{n(g_2)} = 0$. Hence the pair $(g, v)$ obviously satisfies critical link monotonicity but $g$ is not strongly pairwise stable.

3. Networks with critical agents

This naturally leads to the question: What conditions are required to make efficient networks strong pairwise stable under the component egalitarian allocation rule? Interestingly, this leads to a condition on those players occupying a critical position in a network.

**Definition 3.1.** A player $i \in N$ has a critical position in the network $g \in G^N$ if there exists some set of links $h^* \subseteq L(g)$ under the control of player $i$ in $g$ such that, there are at least two distinct players $j_1, j_2 \in N \setminus \{i\}$ who are connected in $g$ and who are not connected in $g \setminus h^*$.

A player occupying a critical position in a network $g$ is referred to as a critical agent in $g$. The set of critical agents in $g$ is denoted by $\mathcal{M}(g) \subseteq N$. A subset $h^* \subseteq L(g)$ of links that a critical agent $i \in \mathcal{M}(g)$ can delete to break up communication within a network $g$ is called the critical link set for that player.

Let $g \in G^N$ be some network with a critical agent $i$ and let $h \in C(g)$ be the non-empty component with $i \in N(h)$. Now we denote by $\mathcal{C}(h \setminus h^*) = \{h_1, h_2, \ldots, h_m\}$ the components obtained from $h$ by deleting a critical link set $h^* \subseteq L(g)$. Clearly, one of these components could be empty, i.e., when $N_0(h \setminus h^*) = \emptyset$. Furthermore, we denote by $\hat{h} \in \mathcal{C}(h \setminus h^*)$ as the component of $h$ with $i \in N(\hat{h})$. Note that if player $i$ herself becomes an isolated node after the removal of $h^*$, then $\hat{h}$ itself is an empty set.

**Definition 3.2.** A pair $(g, v) \in G^N \times V^N$ is disintegration proof if for every component $h \in C(g)$, every critical agent $i \in \mathcal{M}(h)$, and every critical link set $h^* \subseteq L(h)$ for agent $i$ we have that

$$v(h) \geq \sum_{k=1}^{m} v(h_k) \text{ implies that } \frac{v(h)}{n(h)} \geq \frac{v(\hat{h})}{n(\hat{h})},$$

where $C(h \setminus h^*) = \{h_1, h_2, \ldots, h_m\}$ and $\hat{h} \in \mathcal{C}(h \setminus h^*)$ such that $i \in N(\hat{h})$.

\(^3\)In fact, $g_1$ is either a chain network of length four or a four-player star network or a network consisting of a single link. Similarly, $g_2$ is either a network consisting of a single link or the empty network. Note that for any of these networks the value under $v$ is zero.
We first show that disintegration proofness automatically implies critical link monotonicity in a network where all components accrue non-negative value.

**Proposition 1.** Let \( v \in V_+^N \). If \((g, v)\) is disintegration proof, then \((g, v)\) is critical link monotone as well.

**Proof.** Consider any non-empty component \( h \in C(g) \) of the network \( g \) satisfying the above condition and a critical link \( ij \in h \). Denote by \( h_1 \) and \( h_2 \) the two components in the network \( h - ij \) where \( i \in N(h_1) \) and \( j \in N(h_2) \). We have to consider three cases:

**Case A.** \( h_1 = h_2 = \emptyset \). Critical link monotonicity is trivially satisfied.

**Case B.** \( h_1 \neq \emptyset \) and \( h_2 = \emptyset \). Now \( n(h) \geq 3 \), \( n(h_1) = n(h) - 1 \geq 2 \), \( n(h_2) = 1 \) and \( v(h_2) = 0 \). Suppose that \( v(h) \geq v(h_1) + v(h_2) = v(h_1) \). Then from the disintegration proofness condition applied to critical agent \( i \) and the critical link set \( \{ij\} \), it follows that

\[
\frac{v(h)}{n(h)} \geq \frac{v(h_1)}{n(h_1)}.
\]

Further, \( v(h) \geq 0 \) and \( v(h_2) = 0 \) automatically implies

\[
\frac{v(h)}{n(h)} \geq \frac{v(h_2)}{n(h_2)}.
\]

Hence, critical link monotonicity follows.

**Case C.** \( h_1 \neq \emptyset \) and \( h_2 \neq \emptyset \). Here both players \( i \) and \( j \) could be critical agents. Considering player \( i \) as the critical agent with critical link set \( \{ij\} \), disintegration proofness for \( i \) implies that

\[
v(h) \geq v(h_1) + v(h_2) \Rightarrow \frac{v(h)}{n(h)} \geq \frac{v(h_1)}{n(h_1)} \quad (6)
\]

Similarly, considering player \( j \) as the critical player implies that

\[
v(h) \geq v(h_1) + v(h_2) \Rightarrow \frac{v(h)}{n(h)} \geq \frac{v(h_2)}{n(h_2)} \quad (7)
\]

Hence, from Eqs. (6) and (7) it follows that

\[
v(h) \geq v(h_1) + v(h_2) \Rightarrow \frac{v(h)}{n(h)} \geq \max \left[ \frac{v(h_1)}{n(h_1)}, \frac{v(h_2)}{n(h_2)} \right]
\]

which is equivalent to critical link monotonicity.

This completes the proof of the assertion. □

We point out that the value function \( v \) in Example 2.2 is not disintegration proof. Consider the critical link set \( h^* = \{ih, ik\} \) for critical agent \( i \) in the network \( g = \{ij, ih, ik, jm\} \). Severing all links in \( h^* \) results in one non-null component \( h_1 = h = \{ij, jm\} \) and two disconnected players \( h \) and \( k \). Now, \( v(g) = 10 > 9 = v(h) + v(\emptyset) + v(\emptyset) \) but \( \frac{v(h)}{n(h)} = 2 < 3 = \frac{v(h)}{n(h)} \). This illustrates that the reverse of Proposition 1 does not hold.
Proposition 2. If \( g \in G^N \) is efficient relative to a component additive \( v \in V^N \), then \( g \) is strongly pairwise stable with respect to the component-wise egalitarian allocation rule \( Y^{ce} \) if and only if \((g, v)\) is disintegration proof.

Proof. Without loss of generality we restrict ourselves to a network \( g \in G^N \) that consists of a single component, i.e., \( \#I(g) = 1 \), and such that \( g = \emptyset \).

Only if: Suppose \( g \) is efficient relative to \( v \) as well as strongly pairwise stable for \( Y^{ce} \) relative to \( v \). Then \( g \) is SLDP. So for any critical link set \( h^* \subseteq L_r(g) \), it must hold that \( i \) does not wish to sever the links in that set. This requires

\[
\frac{v(g)}{n(g)} = \frac{v(h)}{n(h)} \quad (8)
\]

where \( C(h \setminus h^*) = \{h_1, h_2, \ldots, h_m\} \) and \( \hat{h} \in C(h \setminus h^*) \) such that \( i \in N(\hat{h}) \); which clearly implies that disintegration proofness holds for \((g, v)\).

If: Suppose that \( g \) is efficient relative to \( v \) and that \((g, v)\) is disintegration proof.

We first show that \( g \) must be SLDP. Severing a non-critical link set by any player will only change the value of the component without changing the number of players in that component. By efficiency of \( g \) and component additivity of \( v \), this value is already at a maximum and hence there can be no net gain.

Suppose that some \( i \in M(g) \) in \( g \) severs a critical link set \( h^* \) from \( L_r(g) \). This results in the component set \( C(g \setminus h^*) = \{h_1, \ldots, h_m\} \). Such an action has no benefit for critical player \( i \) because by efficiency of \( g \) and component additivity, we have that

\[
v(g) \geq \sum_{k=1}^{m} v(h_k) \quad (9)
\]

which by disintegration proofness implies that Eq. (8) has to hold. Thus \( g \) is SLDP.

From efficiency of \( g \) it follows that \( v(h) \geq 0 \) for all components \( h \in C(g) \). Hence, by Proposition 1 disintegration proofness implies critical link monotonicity. From Claim 1, network \( g \) is pairwise stable and, therefore, link addition proof. Hence, \( g \) has to be strongly pairwise stable. □

4. Networks without critical agents

We now turn to the study of networks that are always disintegration proof, irrespective of the network value function employed. Formally, a network \( g \in G^N \) is called well-connected if \( M(g) = \emptyset \). The next proposition shows that only well-connected networks are disintegration proof for arbitrary value functions.

Proposition 3. A network \( g \in G^N \) is well-connected if and only if for every network value function \( v \in V^N \) the pair \((g, v)\) is disintegration proof.

Proof. First note that if \( n=2 \) the assertion is trivially satisfied. Next, consider the case when \( n \geq 3 \) and the network \( g \in G^N \) consists of a single non-trivial component, i.e., \( \#I(g) = 1 \). Since \( n \geq 3 \) it is obvious that \( g \) has to consist of at least two links.

4In graph theory, such networks are referred to as 2-connected or bi-connected graphs.
If: Suppose to the contrary that \( g \) has at least one critical agent denoted by \( i \in \mathcal{M}(g) \). We proceed by constructing a network value function \( \nu' \) for which \((g, \nu')\) is not disintegration proof. Note that by definition it has to hold that \( \#L(g) \geq 2 \).

Next, consider a critical link set \( h^* \subseteq L(g) \) such that \( C(g \setminus h^*) = \{h_1, \ldots, h_m\} \) with \( i \in N(h_1) \) and \( n(h) > n(h_1) \). It is clear that since \( g \) consists of at least two links, we can select the critical link set \( h^* \) for critical agent \( i \) in this fashion.

Now select the network value function \( \nu' \) such that \( \nu'(h) = \nu'(h_1) = 1 \) and \( \nu'(h_k) = 0 \) for all \( k = 2, \ldots, m \). Then we have that \( mV(h) = 1 \), \( mV(h_1) = \sum_{k=1}^{m} \nu'(h_k) \), and

\[
\frac{v'(h)}{n(h)} = \frac{1}{n(h)} < \frac{1}{n(h_1)} = \frac{v'(h_1)}{n(h_1)}.
\]

This implies that disintegration proofness is not satisfied for the pair \((g, \nu')\).

Only if: Suppose that \( g \) is well-connected. Since \( \mathcal{M}(g) = \emptyset \), for any network value function \( v \in V_N \) the pair \((g, v)\) is trivially disintegration proof. \( \square \)

Combining Propositions 2 and 3 immediately leads to the following:

**Corollary 4.1.** If \( g \in G_N \) is well-connected as well as efficient relative to some component additive \( v \in V_N \), then \( g \) is strongly pairwise stable for the component-wise egalitarian allocation rule \( Y^{ce} \).

5. **Concluding remarks**

In this paper we reconciled the requirements of efficiency and strong pairwise stability for the component-wise egalitarian allocation rule on the class of cooperative games endowed with a communication network. We found that critical positions instead of critical links have a crucial role in this reconciliation; an efficient network is strongly pairwise stable if and only if these players occupying critical positions in the communication network have no incentive to break up the network.

This result puts the spotlight on these critical positions and the players occupying them. In several economics applications such players are called “middlemen” (Kalai et al., 1978) and their critical role in certain economic activity such as trade and quality control has been confirmed (Biglaisser, 1993; Biglaisser and Friedman, 1994). This role is also at the foundation of the concept of “structural hole” in a social network, seminally introduced by Burt (1992). Therefore, we conclude that our main result calls for a more broad investigation of these critical agents and future research should identify their game theoretic and social network properties more clearly.

**References**


