ANOTHER LOOK AT THE FORWARD-FUTURES PRICE DIFFERENTIAL IN LIBOR MARKETS

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IN LIBOR MARKETS

The difference between forward and futures prices accounted for by the daily settlement is re-examined for one of the most active financial markets in the world, the LIBOR or Eurodollar market. Previous research has observed a difference that may be over- or under-stated as a result of the different pricing conventions in the spot and futures markets. The spot market uses add-on interest while the futures contract design treats the contract as though the spot market uses discount interest. Empirical data are examined and demonstrate the difficulty of accurately measuring this price differential. An alternative test is conducted that estimates the evolution of the term structure using the Heath-Jarrow-Morton model to yield arbitrage-free futures prices. Using a data set consisting of daily LIBOR spot rates of from one to twelve months for about 3,500 days over the period 1987-2000, the difference between Eurodollar forward and futures prices is found to be much smaller than had been previously thought. The difference is negative as it should be but very close to zero. Further tests reveal that more accurate estimates of volatility give even smaller differences but that it is not difficult to obtain satisfactory estimates of volatility for purposes of examining this issue.
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A forward contract is a firm commitment for one party to buy an underlying asset from another party at a fixed price agreed upon up front. No money changes hands at the start, the terms of the contract are customized to the two parties, and no payments are made until the expiration of the contract, at which time the buyer pays the agreed-upon price to the seller, who delivers the asset to the buyer. A futures contract is similar to a forward contract. Its price is negotiated between the two parties, but it has a standard set of terms, is traded on a futures exchange, and is subject to a daily settling of gains and losses, which pass from buyer or seller through the exchange’s clearinghouse to seller or buyer, whichever the case may be. Forward contracts impose upon each party the risk that the counterparty will default, but futures contracts are guaranteed against default by the clearinghouse.¹ Both types of contracts provide that one party agrees to pay a fixed price at the contract expiration and the other party agrees to deliver the underlying asset at expiration. Futures contracts trade in a market and are usually offset before expiration, thereby obviating delivery. Forward contracts are usually held to expiration, but a party can offset the position by entering into the opposite transaction with the same counterparty or offset the market risk of the position by entering into the opposite transaction with a different counterparty, though in that case, any credit risk would remain.² Forward and futures contracts are often used for the same purpose.

An important question that has appeared in the finance literature is whether there is a difference between the price of a forward contract and the price of futures contract. Under the perfect market assumptions and the condition that forward as well as futures contracts are default-free, Cox, Ingersoll, and Ross (1981), Jarrow and Oldfield (1981), and Richard and Sundaresan (1981) identify when a forward-futures price differential exists. Thus, any such difference in the price of a forward contract and the price of futures contract is strictly a result of the different cash flow patterns of the two contracts. Futures contracts, by virtue of their periodic settling of gains and losses, have cash flows during the life of the contract. In contrast, forward contracts have their entire cash flow deferred to the expiration of the contract. When interest rates are non-

¹Futures clearinghouses were first created in the 1920s and have been successful in insuring against default since that time. Some defaults have occurred, but the clearinghouses have always paid the party who holds the claim against the defaulting counterparty. The clearinghouse then has a claim against the defaulting party’s clearing firm, which has a claim against the defaulting party or its brokerage firm.

²Oftentimes forward contracts are described as illiquid, while it is claimed that futures contracts are liquid. This statement is, however, misleading and sometimes completely inaccurate. Many forward markets are characterized by numerous participants, which permits parties holding open positions to offset the market risk of their forward positions. In contrast, some futures contracts are barely traded at all.
stochastic, forward and futures prices are equal. When interest rates are stochastic, futures prices will exceed forward prices if futures prices are positively correlated with interest rates. A positive correlation creates the condition that long positions in futures contracts produce positive mark-to-market cash flows during periods of rising interest rates and negative mark-to-market cash flows when interest rates are falling.\(^3\) Under these conditions, the process of marking a futures contract to market is beneficial to the buyer of the contract over simply settling the contract at expiration. This preference by long parties for futures contracts over forward contracts raises futures prices above forward prices. These points are usually referred to as the Cox-Ingersoll-Ross propositions. Of course, differences in forward and futures prices could exist due to differences in their credit risks, but the Cox-Ingersoll-Ross propositions assume no credit risk in forward contracts. Thus, these propositions deal only with the impact of the cash flow differences when default-free parties are involved.

Whether the forward-futures price differential in reality is non-zero is an empirical question. Unfortunately, the issue is a difficult one to test, and, consequently, there has not been much empirical research on the subject. Futures price data are widely available, but forward price data are somewhat more difficult to obtain.\(^4\) Even when forward price data are available, however, it can be difficult to obtain a sufficient quantity of observations. For example, many futures contracts have only four expirations a year. To compare the forward-futures price differential, it is necessary to match the price of a futures contract, which has a specific expiration, to that of a forward contract with the same expiration. Forward price data are often available only with certain expirations, which can make it difficult to obtain a sufficient number of observations that align with the futures contract expirations.\(^5\) Moreover, measurement errors in the data can greatly interfere with the ability to draw reliable inferences from such tests.\(^6\)

\(^3\)Technically the condition is that futures prices are higher than forward prices if the covariance between futures prices and the price of zero coupon bonds is negative. In addition, futures contracts will have higher prices than forward contracts if the covariance between forward prices and zero coupon bond prices is negative or if the covariance between the spot price and the zero coupon bond price is less than the variance of the zero coupon bond price.

\(^4\)At this point, we should distinguish between actual forward contracts and synthetic forward contracts. An actual forward contract price would be the transaction price for a forward contract entered into between two parties. Such data are virtually never available because the transactions are private. A synthetic long (short) forward contract can be constructed by purchasing (selling) the underlying asset and borrowing (lending). The synthetic forward price and actual forward price should be the same or at least, extremely close, or otherwise there would be an opportunity for an arbitrage profit. Synthetic forward prices are, therefore, constructed from spot prices and interest rates.

\(^5\)Consider the following example, which would apply to many interest rate contracts. A given futures contract has expirations of a specific day in March, June, September, and December. The underlying has a 90-day maturity at the futures expiration. Interest rate forward contracts typically are offered with maturities of 30, 60, 90, …, 360 days. In a given year, the March futures contract price can be matched with the price of a forward contract two times (e.g., in January 60 days ahead and in February 30 days ahead), the June contract can be matched five times, the September contract can be matched eight times, and the December contract can be matched eleven times. Thus if one has 250 days of observations in a year, only 26 are usable. Some of these dates will fall on weekends so even fewer might truly be available. Thus, a researcher may often have to discard over 90 percent of the data.

\(^6\)See, for example, French’s (1983) study of the copper and silver markets.
Indeed, research on the forward-futures price differential is fraught with data difficulties and inconsistencies.

This study provides a new look at the forward-futures price differential, focusing on the Eurodollar or LIBOR market. This market provides a particularly attractive setting for examining this issue. The spot, forward, and futures markets are the most active of all, thereby largely removing the primary concerns that any conclusions one draws might be related to liquidity issues. Another reason for the attraction of using this market is that the inverse relationship between bond prices and interest rates means that since Eurodollar futures prices are negatively related to interest rates, Eurodollar forward prices should unarguably exceed Eurodollar futures prices. Hence, the impact of marking-to-market is not only unambiguous but should be as strong and as significant as one could conceivably expect to find anywhere in financial markets. To find that Eurodollar futures prices exceed Eurodollar forward prices would imply irrational investors or that something is wrong with the tests or data. Of course, any such effect could be statistically significant but economically insignificant, so for all practical purposes, such effects could be ignored and futures and forward contracts could be regarded in many contexts as the same instrument.

We re-examine this issue by looking at previous empirical work, noting a subtle but important problem. We then conduct an empirical test and show the difficulty, if not impossibility, of finding in empirical data the answer to the question of whether Eurodollar forward prices are higher than Eurodollar futures prices, as they should be. We then provide an alternative approach, which blends empirical analysis with the estimation of a term structure model that yields arbitrage-free futures prices that overcomes the subtle problem we see in empirical research on this subject.

Section I examines the two previous empirical studies on this subject and discusses the problems involved in empirically testing this issue. Section II describes the data and the design of the empirical tests that will be done in this study. Section III reports the results of empirical tests that simply compare actual forward and futures prices. Section IV describes how we fit an arbitrage-free binomial tree to forward market data and use that tree to produce arbitrage-free futures prices, which will be free of problems associated with empirical futures prices. Section V reports the results of tests of these hypothetical futures prices in comparison to forward prices. Section VI provides a summary and conclusions.
I. Previous Studies and the Problems of Testing the Forward-Futures Price Differential

Futures contracts on 90-day Eurodollars are (currently) the most active futures contract in the world. The 2002 Annual Report of the United States Commodity Futures Trading Commission (CFTC) gives the volume as over 208 million contracts, with each contract covering $1 million of Eurodollars.\(^7\) Average month-end open interest in 2002 was almost 4.5 million contracts. Thus, in dollar terms, open interest was about $4.5 trillion. The Eurodollar forward market is also one of the most active financial markets. The Bank for International Settlements (www.bis.org) reports that as of December 2002, the global market for interest rate forward contracts, called FRAs, is about $8.8 trillion in notional principal with a market value of $22 billion. U.S. dollar transactions are about a third of that. One of the reasons for the success of the Eurodollar futures contract is the fact that over-the-counter derivatives dealers use the contract to hedge their positions in LIBOR-based interest rate swaps, forwards, and options. Indeed, this market is large, highly successful, and probably the most important market in which to examine the question of whether forward prices equal futures prices.

A. Prior Research

The first empirical study of the forward-futures price differential in LIBOR markets is Meulbroek (1992). Meulbroek’s data consist of daily LIBOR rates for 1-, 2-, 3-, 6-, and 12-month maturities for the period March 1982 to June 1987, a total of 1,232 days. Because of the aforementioned limited availability of data, she infers forward rates for other maturities by fitting a quadratic regression. There is some uncertainty, however, in the manner in which Meulbroek calculates rates. Her equation (1) is correct for the calculation of Eurodollar forward rates, but there is no indication of how she calculates futures rates. Assume, as an example, that on the day of expiration of a futures contract 90-day spot LIBOR is 6%. Price reporting services will state that the contract settled at a price of 100 – 6 = 94. The actual futures price, however, is 100 – 6(90/360) = 98.5. Thus, the terms of the contract specify that the buyer is effectively purchasing 100 Eurodollars at a price of 98.5. This result can imply a compound rate of \((100/98.5)^{365/90} – 1 = .063212\), or \((100/98.5)^{360/90} – 1 = .062319\) if a 360-day year is assumed. Alternatively, since the spot market for Eurodollars calculates interest in the add-on manner, the rate could be found differently. A Eurodollar time deposit of 100 would grow to \(100(1 + .06(90/360)) = 101.50\) for an effective rate of \((101.50/100)^{365/90} – 1 = .062242\) or \((101.50/100)^{360/90} – 1 = .061364\) using 360 days. Thus, the “futures rate” could be considered to be 6%, 6.3212%, 6.2319%, 6.2242%, or

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\(^{7}\)The CFTC’s fiscal year is the U.S. government’s fiscal year, which ends in September. Thus, this volume figure covers the period October 1, 2001 through September 30, 2002.
6.1364%. Except for the third and fourth numbers, the variation in basis points is substantial. It is not even clear which is the correct or best number.

Meulbroek fits the term structure based on LIBOR spot rates, which are handled correctly and unambiguously, but some of the results (e.g., her Table III, p. 388) discuss the statistical properties of forward and futures rates so the issue of how the futures rates are defined is important. Fortunately, most of her remaining results are in terms of prices so there is no ambiguity. She tests several related questions, but the primary one of our interest is the sign and magnitude of the forward-futures price differential. As noted, the Cox, Ingersoll, Ross propositions state that futures prices will exceed corresponding forward prices if the correlation between futures prices and interest rates is positive.

Meulbroek calculates the futures price minus the forward price, so she should obtain negative values. Her results show for the nearby contract that the average differential is –0.0002 for a contract par value of $1. For the second nearby contract, the differential is –0.0024. For the third nearby contract, the differential is –0.0048. As we show later, however, it would be important to know if any of the sample observations are positive, because there is reason to believe that some, if not many, probably are.

A more recent empirical study is Grinblatt and Jegadeesh (1996) who examine daily data over the period 1982-1992 for spot maturities of 1-, 2-, 3-, 6-, and 12-months. Because essentially all of their results are couched in terms of rate differences instead of price differences, the question of how the futures rate is defined is more severe for this paper. While they show that they correctly relate Eurodollar spot prices to Eurodollar spot rates, they state that $F(s,\tau;0)$ is the “Annualized Eurodollar futures rate at $t$ for the interval $s$ to $\tau$. ($t$ is omitted when $t = 0.$)” (p. 1501). They proceed (p. 1502) to state that the futures rate is given as $F(s,s+0.25;0) = 1 – \text{Futures Price}_t/100$ (their equation (2)). As in the example we gave above, this would suggest that 6% is the rate they would use. As we stated above, it is not clear that this is the correct value for comparison purposes. Hence, an argument could be strongly made to avoid rate comparisons at all costs. Price comparisons are clearly preferred.

To generate a sufficient number of matched observations, Grinblatt and Jegadeesh proceed to fit two interpolated term structures. One term structure fits a cubic spline to the available futures rates. Of course what one obtains from this procedure would be unclear, given the aforementioned problem in defining the appropriate “futures rate.” The other method fits a cubic spline to the spot LIBOR rates, which would be clearly appropriate and consistent with Meulbroek. All of their remaining results are based on rate differences obtained from these estimated term structures. Half of the tests are based on a comparison of LIBOR forward rates
compared to date-matched futures rates obtained from the interpolated futures term structure, and half are based on a comparison of futures rates compared to date-matched forward rates obtained from the interpolated term structure. In all cases, however, rates are compared to rates.

Their forward rate is defined as \([(1/\text{forward price}) - 1]\times 360/90, which is correct for the LIBOR market. Again, their futures rate is \(1 - \text{futures price}\). From these results, we would like to be able to convert their rate differences into price differences. Unfortunately, algebraic rearrangement of these expressions indicates that this is not possible without knowing the level of the forward rate. Using the data set we describe later in this paper, we estimate the average forward rate for a period that overlaps the Grinblatt-Jegadeesh study and the present study, 1987-1992. Using this information with their quoted results, we find that the price differences in basis points per $1 par are estimated at

<table>
<thead>
<tr>
<th>Maturity</th>
<th>Average</th>
<th>Range</th>
</tr>
</thead>
<tbody>
<tr>
<td>(using the futures term structure)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>0 – 3 months</td>
<td>-3.2</td>
<td>-0.5 to -7.2</td>
</tr>
<tr>
<td>3 - 6 months</td>
<td>-3.2</td>
<td>-0.7 to -7.5</td>
</tr>
<tr>
<td>6 - 9 months</td>
<td>-6.2</td>
<td>-3.6 to -10.2</td>
</tr>
<tr>
<td>(using the spot/forward LIBOR term structure)</td>
<td></td>
<td></td>
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<tr>
<td>0 – 3 months</td>
<td>-2.3</td>
<td>-0.2 to 6.5</td>
</tr>
<tr>
<td>3 - 6 months</td>
<td>-3.2</td>
<td>-0.7 to -7.5</td>
</tr>
<tr>
<td>6 - 9 months</td>
<td>-4.6</td>
<td>-0.2 to -8.7</td>
</tr>
</tbody>
</table>

These differences are smaller than those reported by Meulbroek, but hers are exact and these are rough estimates. As we noted, Meulbroek did not report maximum and minimum values and gave no indication if any differences were positive, but Grinblatt and Jegadeesh’s results do show some negative rate differences, which lead to positive price differences (0-3 months in the spot/forward LIBOR term structure above). These negative rate differences are particularly noticeable in their graphs in their Figure 1 (pp. 1504-1506). Such differences suggest that investors are irrational. and do not have a preference for the settlement at expiration of forward contracts, as Cox-Ingersoll-Ross prove they would. More importantly, these negative rate differences that lead to positive price differences bias the overall average toward zero, leading to a downward bias when trying to ascertain the true differences between futures and forward prices or rates.

A second stage of the Grinblatt-Jegadeesh paper involves the estimation of theoretical differences in forward and futures rates using the Cox-Ingersoll-Ross (1985) and Vasicek (1977) term structure models. These theoretical differences are compared to actual differences. Given
the large gap between the two, Grinblatt and Jegadeesh search for alternative explanations in the form of market imperfections or market participants simply mispricing futures. They conclude that the mispricing is due to an unexplained lack of arbitrage activity. Their data set is used to estimate the parameters of both models, but an important limitation of both of these models is not addressed in their study. It is well known that a model like the Cox-Ingersoll-Ross or Vasicek model is arbitrage-free only within the model itself but not external to the model. Both models produce the term structure as an output. Neither model is calibrated to the current term structure. Hence, anyone trading in an actual market with one of these models is susceptible to arbitrage losses to other parties. Fully arbitrage-free models like Heath-Jarrow-Morton (1992), Ho-Lee (1986), or Black-Derman-Toy (1990) fit the current term structure and prohibit all arbitrage in the market.

B. Complications from the Settlement Feature of Eurodollar Futures

An even greater concern and a central theme of the present paper is that the tests of Meulbroek and Grinblatt-Jegadeesh ignore a critical characteristic of the Eurodollar futures market. Suppose the futures contract is being priced on day \( t \) and expires on day \( T \). The futures price is denoted \( F(t,T) \). Now move ahead to time \( T \), the expiration. A 90-day Eurodollar time deposit, the instrument on which the futures contract is based, has a rate of \( L(T,T+90) \). At expiration, the present value of $1 ninety days later is

\[
B(T, T+90) = \frac{1}{1 + L(T, T+90)(90/360)}.
\]

This value can be viewed as the spot price at expiration. The futures price at expiration is defined by the terms specified in the contract by the Chicago Mercantile Exchange:

\[
F(T, T) = 1 - L(T, T+90)(90/360).
\]

Note that the spot instrument, whose rate drives the futures price, is in the form of an instrument in which the interest is added on to the amount invested. That is, \( B(T,T+90) \) dollars invested in a Eurodollar time deposit grows to $1 by the factor \( 1 + L(T,T+90)(90/360) \). The futures contract is priced, however, as though the underlying were a discount instrument, such as a U. S. Treasury bill.\(^8\)

The theory underlying the determination of futures prices typically ignores the daily settlement feature and proposes a hypothetical transaction consisting of buying the spot

\(^8\)It appears as if the Eurodollar contract design was cloned from the U. S. Treasury bill futures contract, which had begun trading six years earlier. The Treasury bill contract was successful and traders were familiar with it. Moreover, this particular design permits a contract with $1 million face value to change in value by $25, given a one basis point move in the underlying rate. This feature facilitates the mental effort required to trade quickly. Ironically, the Treasury bill contract is no longer very actively traded, while the Eurodollar contract is the most active U. S. futures contract. In fact, for every Treasury bill futures contract that trades, over 20,000 Eurodollar futures contracts trade.
instrument and hedging its future delivery by selling a futures contract. Consider the following strategy:

Time $t$

Buy a Eurodollar time deposit expiring at time $T + 90$ by investing $B(t, T+90)$.

Sell a Eurodollar futures expiring at time $T$ for price $F(t, T)$

Time $T$

The Eurodollar time deposit is worth

$$B(T, T + 90) = \frac{1}{1 + L(T, T + 90)(90/360)}.$$ 

The futures price is

$$F(T, T) = 1 - L(T, T + 90)(90/360).$$

The payoff of the futures is $F(t, T) - F(T, T)$ or

$$F(t, T) - (1 - L(T, T + 90)(90/360)).$$

The value of the overall position of the futures contract and the Eurodollar time deposit is

$$\frac{1}{1 + L(T, T + 90)(90/360)} + F(t, T) - (1 - L(T, T + 90)(90/360)).$$

To derive an arbitrage-free futures price, this transaction, which is typically called cash-and-carry, must be risk-free. In this case, however, it is not. In simple terms, the futures price and spot price do not converge. Moreover, there is no way to alter one’s holdings of either the Eurodollar time deposit or Eurodollar futures to offset the risk.\(^9\) Hence, it is not possible to price the futures contract using this approach. If the instrument were a forward contract, convergence would occur and lead to the standard cost of carry pricing formula. The convergence of futures and spot prices is a common feature of most futures contracts and plays a critical role in the Cox, Ingersoll, and Ross propositions. The effect of this non-convergence of futures and spot prices is not considered in the Meulbroek and Grinblatt-Jegadeesh tests.

This unusual settlement feature of the Eurodollar futures contract has, however, been discussed by Sundaresan (1991), who provides an adjusted model based on the general

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\(^9\)The problem boils down to the simple equation, $1/(1+y) - (1+y) = 0$. There is no solution except $y = 0$, which would imply an interest rate of zero. Alternatively, if a weight is applied to one term, the solution for the weight is a function of the random interest rate. Thus, for a simple cash-and-carry transaction (buy spot, sell futures), there is no ex ante perfect hedge. It is, however, still possible to price the futures. As we discuss later, any futures contract is the expectation of the futures price at a later date, where expectations are taken using the martingale probability measure. We shall do this within a hypothetical setting using the Heath-Jarrow-Morton (1992) model, but it can be demonstrated without resorting to any particular model of the term structure, provided simply that the term structure is arbitrage-free.
equilibrium term structure model of Cox, Ingersoll, and Ross (1985).\footnote{Even though it was published in 1991, Meulbroek (1992) does not cite the Sundaresan paper. Grinblatt and Jegadeesh (1996) cite it but make only a casual reference to it as having addressed the pricing of Eurodollar contracts. They do not discuss the primary theme of the Sundaresan paper, the complexity added by the settlement feature.}

His adjustment accounts for the feature, but requires a numerical solution to a partial differential equation. Sundaresan computes forward-futures price differentials based on a range of assumed inputs and finds an average differential of less than -0.0002 per $1 par for contracts expiring in 90 days and -0.0006 for contracts expiring in 180 days.\footnote{Some adjustments are required to interpret Sundaresan’s findings. Consider his Table 2 on p. 419. For the first case, the futures price is given as 92.74 and the forward price is given as 92.78. These are quoted prices, which are simple linear prices, based on a subtraction of a rate from a par value of 100. The actual futures price would be found as 100 – (100 – 92.74)(90/360) = 98.185. The forward price would be equivalent to 100 – (100 – 92.78)(90/360) = 98.195. This is a difference of 0.01. Using $1 par as standard, the difference is 0.0001. Sundaresan quotes the difference as 4.66 basis points, which is based on the quoted prices of 92.74 and 92.78, carried out to more decimal places, and multiplied by 100. To adjust for this effect, the results in the final column in Sundaresan’s Tables 2 and 3 should be multiplied by a factor of 1/(4*10,000).} These differences are, however, not based on actual data, but on assumptions about the parameters of the stochastic process driving the spot rate as well as an assumed level of the spot rate. Sundaresan does provide some limited empirical evidence based on 13 observations, another clear case of the data availability problem, and are also subject to the settlement problem. He finds that the difference averages $0.00125 per $1 par for 90-day contracts.\footnote{The forward-futures price differential has also been studied in other markets, some of which are interest-rate instruments (usually Treasury bills) and some of which are other types of assets. See Elton, Gruber, and Rentzler (1984) Kolb and Gay (1985), Capozza and Cornell (1979), Cornell and Reinganum (1981), Kawaller and Koch (1984), Gendreau (1985), Park and Chen (1985), and Allen and Thurston (1988). In fact, virtually any study of the pricing of futures contracts in relation to spot prices treats the futures contract as though it were a forward contract and is, therefore, a joint test of pricing efficiency and the forward-futures price differential.}

Thus, Meulbroek’s results, using empirical data over a five-year period, show a differential of -0.0002 to -0.0048 per $1. Grinblatt and Jegadeesh, using empirical data on rate differences over a 10-year period (but converted to price differences over a five-year period) show differences of -0.0002 to -0.0010 per $1 with some results being irrationally positive. Sundaresan, using the Cox, Ingersoll, and Ross model and hypothetical inputs, shows a differential of -0.0002 to -0.0006, and -0.00125 using empirical data.\footnote{Technically, Meulbroek’s differences are slightly less negative than stated in comparison to Sundaresan’s because she expresses her difference as the log of the futures price over the forward price. For negative differences, the log difference will be slightly more negative. This effect is, however, fairly minor.} Sundaresan’s work, which accounts for the settlement difference, is based on numerical inputs into a model that does not fit the current term structure. His limited empirical results are also subject to the settlement problem. There is clearly a need for additional research that uses a sufficient amount of quality data and addresses the settlement problem.

To get an idea of the magnitude of this settlement effect at expiration, Figure 1 shows the difference between forward and futures prices at expiration for a $1 par contract for a reasonable
range of LIBOR values of 1 – 10%. For example, if LIBOR is 6%, the forward price is $1/(1 + 0.06(90/360)) = 0.98522167, the futures price is 1 - 0.06(90/360) = 0.9850, and the difference is 0.00022167 or 2.2167 basis points. We see that the price differential is positively related to the magnitude of LIBOR and increases at an increasing rate with LIBOR. The settlement effect accounts for up to 10 basis points of the price differential at expiration, which is equivalent to $1,000 for a standard $1 million eurodollar futures contract. This amount is all the more significant considering that the initial and maintenance margins on the futures contract are less than $1,000.

While Figure 1 shows the impact of the expiration settlement effect without considering the effect of the daily marking-to-market, our interest is in the impact of the daily marking-to-market without the expiration settlement effect. We have reason to believe that not accounting for the settlement effect could substantially bias any conclusions we might draw on the relationship between forward and futures prices that arises strictly from the different daily cash flow patterns in the two contracts. Previous research has either failed to account for the expiration settlement effect or has done only a simple analysis using parameters chosen by the researcher. Clearly there is a need to analyze the issue using market data, while accounting for the settlement effect bias.

A separate but related body of research addresses a similar issue. The difference between forward and futures prices is oftentimes attributed to an effect known as convexity. Burghardt and Hoskins (1995a, 1995) and McDonald (2003) describe the convexity effect as arising out of a timing difference in a hedge transaction. Using McDonald’s framework, suppose at time 0, a party anticipates borrowing at time t, with the loan to be repaid at time t + τ. The borrower hedges the interest rate risk using a futures contract expiring at time t. To avoid any effects arising from marking-to-market, assume that the entire payoff of the futures occurs at its expiration t. It is easily shown that a perfect hedge is not possible, because to hedge the interest rate on the loan, the futures payoff realized at t must be compounded to t + τ at a rate unknown at time 0. If this rate were known at time 0, the size of the position could be adjusted to provide a perfect hedge (a process known as tailing), but if that were the case, it would not be necessary to hedge. In the Appendix, we show that this problem is unrelated to the expiration settlement effect by demonstrating that if a borrower issues a discount note at LIBOR and hedges using the Eurodollar futures, settling as it does in the discount manner, a perfect hedge is still not possible. It should be noted that the hedging of an loan with add-on interest using a futures contract in which the interest factor is applied in an add-on manner does lead to a perfect hedge. Hence, the case of discount interest, in which the interest is taken out in advance, prevents perfect hedging.
The case of add-on interest, whereby the interest is paid at maturity, does lead to a perfect hedge. Add-on interest, therefore, neutralizes the timing problem. Thus, the so-called timing problem is distinct from the problem arising from the fact that the Eurodollar spot market settles one way and the futures market settles in another.

McDonald also mentions (p. 215) that marking-to-market is another source of the convexity bias. Gupta and Subrahmanyam (2000) also address the issue in their examination of the difference between observed swap rates and swap rates obtained using the futures term structure. They state that “The differences between futures and forward rates, which leads to the convexity bias in interest rate swaps, are potentially attributable to the marking-to-market feature of futures contracts.” (p. 246). Of course, this result is well-known from the classic Cox-Ingersoll-Ross paper.

Thus, we see that there are three potential sources of a forward-futures price differential: the daily marking-to-market of futures contracts, the timing difference between the expiration of the futures and the maturity date of the underlying, and the difference in the difference in how the spot market settles (add-on interest) and the futures market settles (discount interest). It is the latter effect that is of our concern.

This study re-examines the issue of the forward-futures price differential in two ways. Using a new and larger empirical data set, we examine the actual differences, showing how real data can lead to misleading and inconsistent results. We then examine the differences using estimated futures prices that are based on an arbitrage-free model of the term structure. This aspect of the study blends real-world data and computational results obtained by creating hypothetical futures prices that conform to the no-arbitrage condition imposed on the actual term structure. In doing so, we control for the unique settlement feature of Eurodollar futures contracts.

II. Data and Test Design

The tests reported here use data from the London Eurodollar market. The British Bankers Association (BBA) samples a set of 16 London banks and establishes an official LIBOR at 11:00 a.m. London time each business day. This rate is based on the interquartile range of quotes of the sample banks. The BBA publishes the data on a daily basis on its web site www.bba.org.uk. The data set begins in 1987 and consists of spot LIBOR for Eurodollar deposits of one to 12 months in increments of one month.\textsuperscript{14} Thus, the data are quite granular.

\textsuperscript{14}The Eurodollar market assumes 30 days in a month when quoting rates, but interest calculations typically use the actual number of days.
allowing a fit of a fairly detailed term structure of up to one year, for every day since 1987. For this study, we use data over the period of 1987 through 2000, a total of over 3,500 term structures, so this study set covers a longer time period than other studies.

We obtain data on the Eurodollar futures contract of the Chicago Mercantile Exchange. The data set was purchased from the Institute for Financial Markets, which is the designated supplier of Chicago Mercantile Exchange data. From the its beginning in 1982 through 1994, the contract was available with expirations in March, June, September, and December. Starting in 1995, a November expiration was added, and the following year, expirations of each month became available. These expirations go out a different number of years, depending on the year in the period 1982-2000. In recent years, these have gone out as far as ten years. Given the limitations of the forward market data, however, we are interested in expirations of less than one year. The availability of these monthly expirations, however, provides a greater number of observations than in previous studies, helping to address one of the problems of earlier empirical studies of this type.

The empirical part of this study will compare forward prices calculated from the BBA LIBOR data with futures prices from the CME. This examination will require a careful alignment of contracts and dates, which will be discussed in the next section. One other concern, however, is that there is the potential for nonsynchronous data. The BBA LIBOR rates are established at 11:00 a.m. London time. The futures prices are settlement prices as of the close. This difference in time can be problematic. Interestingly, however, the actual spot market for Eurodollars in London is itself not synchronized with the futures market in Chicago. The BBA establishes the official daily LIBOR around 11:00 London time and publishes it soon thereafter. The futures contract begins trading in Chicago at 7:20 a.m. central time and closes at 2:00 p.m. The official LIBOR is not updated for another day, though unofficial quotes are available from various services. Nonetheless, the Chicago futures market continues to trade until well after the London banks have closed. Thus, in reality the two markets themselves are not synchronized, which can cause further problems in trying to test the relationship between forward and futures prices.

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15 Technically, the expirations available for trading are March, June, September, and December, plus the current month and the next month. Therefore, there will be a contract expiring each month of the year from 1996 onward. The exact expiration day is the second London business day prior to the third Wednesday of the month.

16 It is not clear whether Meulbroek and Grinblatt-Jegadeesh had synchronous data. Meulbroek states that the futures prices were obtained from Goldman Sachs, which were evidently end-of-day prices. Her spot rates were the “latest available from Telerate at the close of the futures market.” Thus, they seem relatively synchronized. Grinblatt and Jegadeesh obtain their futures price data from the CME and the spot data from DRI. There is no indication of whether they are synchronized. It is likely they are not, since the DRI data were stated as being averages from several banks, obtained from Reuters. In all likelihood, these were the 11:00 a.m. quotes.
Another concern is that futures prices are affected by transaction costs, taxes, and bid-ask spreads that differ from those of forward prices extracted from the LIBOR term structure. They, too, are affected by these factors, but to a different extent.\footnote{As noted in Section 1A, Grinblatt and Jegadeesh address some of these market imperfections and conclude that they are insufficient to affect their observed rate differentials to any great extent.}

Finally, we should note that a test of this sort using futures price data would suffer from the same problems of the Meulbroek and Grinblatt-Jegadeesh studies. That is, the futures contract is based on the CME’s design of the futures contract as a discount instrument, while the underlying Eurodollar is an add-on instrument. We shall attempt to address this problem by adjusting the forward prices so that they reflect the prices of instruments constructed from the same rates, but based on the discount procedure used in the futures market.

While the empirical study is conducted to determine if a different, perhaps better, data set can resolve the question, it remains that the settlement feature will corrupt our results. Thus, we need an alternative approach that will be free of the settlement feature problem and the problem of non-synchronicity. This study estimates input parameters from empirical data and fits the term structure to the more general Heath, Jarrow, Morton (1992) model. The Heath-Jarrow-Morton model has the advantage of requiring the estimation of only one general parameter, the term structure of volatility. In addition, the Heath-Jarrow-Morton model is arbitrage-free in that it fits the current term structure of interest rates. The Cox-Ingersoll-Ross model provides the term structure as an output and requires as inputs the long run spot rate, the volatility, and the mean reversion parameter. It does not guarantee that an investor could not earn an arbitrage profit trading with current prices. Therefore, we believe that the Heath-Jarrow-Morton model is a superior approach to fitting the term structure.

The Heath, Jarrow, and Morton model (henceforth, HJM) assumes that the uncertainty in the term structure is driven by an arithmetic diffusion process in the forward rates. HJM is consistent with the current term structure and the volatilities of forward rates. For certain volatility structures, closed-form solutions for the prices of bonds and derivatives are available.\footnote{See, for example, Jarrow and Turnbull (2000, Chs. 15-17) who rely on the assumption that the volatilities of forward rates are mathematically related by an exponentially dampening function.}

For arbitrary volatility structures, the model is usually fit using a multinomial tree. The HJM model can accommodate any number of risk factors that drive the forward rates, but the more factors used, the more complex and computationally intensive is the model. We shall limit our analysis to a single factor, which will be described and justified later.

III. Empirical Tests Using Actual Data
To compare Eurodollar futures prices with Eurodollar forward prices, we must first determine which futures prices will match up with which forward prices. Recall that we have spot prices for 30, 60, 90, …, and 360 days. The Eurodollar time deposit underlying the futures contract has a 90-day maturity. Thus, we can construct forward prices on 90-day Eurodollar time deposits for 30 days ahead, 60 days ahead, 90 days ahead, and so forth to 270 days ahead. We need to determine on which days there is a corresponding futures contract expiring in 30, 60, 90, …, and 270 days. We follow this rule very strictly and avoid bumping any days forward or backward. For example, if the matching contract would require a weekend quote, it would be tempting to bump it forward or backward one day, but we do not do so. The amount of data available provides a reasonable sample size so we do not distort the results with one to two more or less days to maturity in some cases. In all cases our interest calculations are based on the ratio “actual/360”, which is the custom in the Eurodollar market. It so happens that all contracts we construct have 30, 60, 90, etc. days until expiration. Over the 1987-2000 period, we are able to obtain 457 matching quotes with maturities of one to nine months, and this would seem to be a sufficient sample size.

The results are summarized in Table 1. As stated previously, prices are based on a par value of $1. All of the average differences for the various maturities are negative and statistically significant, as we might think they would be. The averages range from less than -0.0001 to -0.0011. Note, however, in the last column that we obtained anywhere from 17.9 to 30.3 percent of our observations greater than zero. This result means that the futures price exceeds the forward price and is inconsistent with the Cox-Ingersoll-Ross proposition that the forward price should exceed the futures price since interest rates and Eurodollar futures prices are inversely related. It should be noted that the Cox-Ingersoll-Ross result does not require any restrictions on preferences. It is a strong result, requiring only mild conditions.

Although the magnitudes of the differences are small, the presence of a large number of positive differences is disconcerting. Imagine a simple case of just two observations, one a large positive difference and one a large negative difference of the same magnitude. The average would be zero but the positive difference half of the time would certainly be inconsistent with rational investors.

These results, however, are contaminated by the fact that the LIBOR time deposit is priced as an add-on instrument, while the Eurodollar futures contract settles as a discount instrument. Perhaps this problem is the source of the irrational positive differences we observe. In an attempt to adjust for this effect, we calculate forward prices as though the underlying
LIBOR time deposits were discount instruments. We refer to this price as the “adjusted forward price.”

The differences between the futures prices and the adjusted forward prices are shown in Table 2. We observe that while the average differences are negative for forward contracts maturing in five to nine months, the average differences are positive for contracts maturing in one to four months. The positive average differences for one- and two-month contracts are significantly different from zero, and the negative average differences for six-, seven-, and nine-month contracts are significantly different from zero. Note also that the percentage of observations greater than zero ranges from 33.9 to almost 95 percent. We can do a rough statistical test for whether the incidence of positive observations is significant. It is not possible to design a statistical test in which the null hypothesis is that all observations are negative. We can, however, state that if all observations are negative, then the sample mean should be negative. If the sample mean is not statistically negative, it would cast doubts on these numbers. A simple t-test applied to the means in Table 2 reveals that the means for expirations of one through five months and also eight months are not significantly negative.\(^{19}\)

Thus, in spite of an attempt to correct for the settlement issue, the problem is even greater. The average differences remain small, but we really cannot trust these results. If futures prices exceed forward prices, investors are irrational.

It is tempting to say that the non-synchronicity of the markets and the data might be the explanation, but there may be more to it than that. We know that because of the differences in the settlement procedures in the two markets, forward contracts and futures contracts that do not converge to each other at expiration are not perfect substitutes, even in a world of constant interest rates.\(^{20}\) It may be the case that empirical data are simply not capable of providing the answer to this important question. As such, we turn to an alternative approach, one that blends empirical data with numerical estimates of empirical prices under idealized conditions.

### IV. Estimating the Forward-Futures Differential Using Blended Data

#### A. Fitting an Arbitrage-Free Tree to Empirical Data

Because of the problems of straightforward estimation of the forward-futures price differential with empirical data, we turn to an alternative approach. We would like to be able to observe prices in an idealized setting, one free from any differences in settlement procedures and where prices are fully synchronized. In a sense, this is what Sundaresan did, using hypothetical

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\(^{19}\)The same test applied to the means in Table 1 reveals that all are significantly negative.

\(^{20}\)Of course, in a world of constant interest rates, we would have to be referring to forward and futures contracts driven by something other than interest rates.
input parameters and the Cox-Ingersoll-Ross model. Also, this is what Grinblatt and Jegadeesh attempted using the Cox-Ingersoll-Ross and Vasicek models with parameters estimated from the data. But these input parameters may not be realistic, may not span a sufficiently wide range of possible values, and the Cox-Ingersoll-Ross and Vasicek models do not fit the current term structure (and, thus, admit arbitrage when used in practice). Also, Grinblatt and Jegadeesh do not account for the settlement feature. So results based thereon can provide only a limited if not biased picture of the possible differences between futures and forward prices.

The approach used here assumes that the LIBOR data contain the fundamental information necessary to capture the term structure. Using the HJM model, we fit an arbitrage-free binomial tree to the term structure, using the actual LIBOR data and volatilities estimated from the time series of the LIBOR rates. This procedure can be used to extract futures prices that are theoretically consistent with the LIBOR term structure and interest rate volatilities, conditional upon the HJM model, and the functional form of the model we choose. These prices will, of course, be free of any problems of non-synchronicity.

The forward prices are calculated from the LIBOR data and are not specific to the choice of term structure model and the estimates of interest rate volatility. If, however, we calculate futures prices the standard way using the LIBOR quotes, we would still face the problem that the futures contract settles as though the underlying is a discount instrument, while the underlying instrument is actually an add-on instrument. We address this problem by constructing a hypothetical futures contract that settles based on the add-on method that is used in the spot market. We do not claim that this is the actual Eurodollar futures price. It is the price that should exist if the futures contract were designed to settle at expiration exactly as the spot instrument is priced. Thus, this is a laboratory experiment that blends real world data that have the essential characteristics of Eurodollar interest rates movements with arbitrage-free prices that would exist in such a market. In so doing, we can more accurately gauge the effect of the different cash flow streams of forward and futures contracts on their prices.\footnote{It is tempting to also do this procedure in the opposite direction. Given the LIBOR rates, suppose we estimate forward prices by constructing an artificial Eurodollar spot instrument, instead of an artificial Eurodollar futures contract, based on the discount method, rather than the add-on method. We could then compute futures prices based on the discount method, as is done in practice, and the comparability issue should be taken care of. The problem with this approach is that the raw information set is the set of spot and forward prices, not the rates. Prices are determined in a competitive market. Since prices represent present values of future claims, they reflect time preferences and expected inflation. Interest rates are simply transformations of prices. If the convention in the Eurodollar market was changed overnight from add-on to discount, Eurodollar spot prices would not change. Thus, it is imperative that the true Eurodollar spot prices be used. These prices can be obtained only by using the add-on method with the LIBOR rates provided by the BBA.}

Given the granularity of the data set, we can, without approximation or interpolation, fit an HJM binomial tree with time steps spaced one month apart out to 11 months. The raw data are
the LIBOR rates provided by the British Bankers Association for 1, 2, 3, ..., 12 months. We denote these as \( L(0,1), L(0,2), L(0,3), \ldots, L(0,12) \). From these we can derive the prices of zero coupon bonds. In general for maturity, \( i \), the price of a LIBOR-based zero coupon bond whose maturity is an integer multiple of 30 at time 0 is\(^{22}\)

\[
B(0,i) = \frac{1}{1 + L(0,i) \left( \frac{30i}{360} \right)}.
\]

The values \( B(0,1), B(0,2), \ldots, B(0,12) \) make up the term structure to which the HJM model will be fit. The forward price based on the term structure at time 0 for a one-period bond starting at time \( i \) is denoted as \( B(0,i,i+1) \) and found as

\[
B(0,i,i+1) = \frac{B(0,i+1)}{B(0,i)}.
\]

The HJM model is based on continuously compounded rates, so we first require the one period spot rate, \( f(0,0) \), which is obtained as\(^{23}\)

\[
f(0,0) = -\ln B(0,1) \left( \frac{365}{30} \right).
\]

The forward rates are found as

\[
f(0,i) = -\ln B(0,i,i+1) \left( \frac{365}{30} \right).
\]

The set of rates \( f(0,0), f(0,1), \ldots, f(0,11) \) constitute the rates that will evolve according to a binomial tree fit to be consistent with the stochastic process of an HJM model and the absence of arbitrage opportunities.\(^{24}\)

The tests in this paper are based on a one-factor HJM model. While the limitations of a one-factor model might raise some questions, empirical results of Litterman and Scheinkman (1991) and Chapman and Pearson (2001) shows that about 88% of the variation in the term structure can be explained by one factor, which is generally thought to be the level of the term structure.\(^{25}\) Moreover, Dybvig (1997) concludes that a single factor HJM-type model should be

\(^{22}\)As described in the beginning of this section, all of our LIBOR-based instruments will have a maturity of an integer multiple of 30 days.

\(^{23}\)Note that we use 360 days when working directly with LIBOR rates and 365 days when computing the continuously compounded rate. The use of 360 days with LIBOR rates is the accepted market convention. When converting to continuously compounded rates, any reasonable convention can be used. Typically the process of continuous compounding or discounting uses the formula Future value = Present value * exp (r*time) where time is in years and measured with a full 365 days.

\(^{24}\)With only 12 spot rates, we can calculate forward rates only up to month 11.

\(^{25}\)The first factor in term structure models is typically viewed as the level of the term structure. The second is considered to be the slope, and the third is called the curvature.
acceptable for the short end of the term structure. The present study is exclusively focused on the short end of the term structure, specifically maturities less than one year.

One obvious question is whether the futures prices we obtain are sufficiently representative of actual futures prices. The estimated futures prices are based on marking-to-market on a monthly basis. It is not possible to fit a term structure that is marked to market daily, as in practice, without having a term structure of bonds with expirations of each day for the life of the futures. No such bonds exist. Moreover, as Heath, Jarrow, Morton, and Spindel (1992) note, their model is considered accurate with usually no more than seven time steps. Their results are all the more relevant to this study, because they are pricing instruments with up to five years to maturity. We are pricing instruments with nine months maturity, using nine time steps. Wall Street firms are known to trade with binomial trees in which the time step is more than one day. Since Eurodollar futures are widely used as a hedging instrument for swaps and interest rate options, it is apparent that a model such as the one estimated here is accepted as a measure of the term structure and the prices of futures contracts.26

B. Forward Rate Volatility Inputs

We shall require volatilities for these forward rates. We denote a given volatility as \( \sigma(i,j) \), which represents the standard deviation at time \( i \) of the forward rate applicable to a one-period bond starting at time \( j \). When fitting the initial tree, time \( i \) is time 0. Thus, we shall require an initial set of forward rate volatilities, \( \sigma(0,1) \), \( \sigma(0,2) \), \ldots, \( \sigma(0,11) \). In fitting the tree, these initial volatilities are used to reflect the movement in forward rates for the first time step. Then for the second time step, we would require another set of volatilities, \( \sigma(1,2) \), \( \sigma(1,3) \), \ldots, \( \sigma(1,11) \). For the third time step, we would require the volatilities, \( \sigma(2,3) \), \( \sigma(2,4) \), \ldots, \( \sigma(2,11) \). This would continue until the entire tree is fit. This formulation would create the tree for the first observation in the data set. For the next observation, a new set of LIBORs the next day, we would fit a new tree. This would continue for the approximately 3,500 days of the data set.

We estimate the volatilities of the forward rates using data from the entire period 1987-2000. We compute the one-period-ahead forward rate volatility, the two-period ahead forward rate volatility, and so forth. These are used as our estimates of \( \sigma(0,1) \), \( \sigma(0,2) \), \ldots etc. For a given day’s term structure tree, we use the estimate \( \sigma(0,i) \) as the estimate \( \sigma(1,i) \). Thus, the volatility of the forward rate 11 months ahead is the volatility of that same rate 10 months ahead, 9 months ahead, etc. This assumption enables the tree to recombine and avoid the massive number of calculations required in a non-recombining tree. This assumption should not be confused with

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26 At the other extreme, the Grinblatt-Jegadeesh results are estimated using the continuous-time Cox-Ingersoll-Ross and Vasicek models and make the assumption of continuous marking-to-market.
the assumption of constant volatility. A constant volatility HJM model is equivalent to the Ho-Lee (1986) model and treats all forward rates as having the same volatility.\textsuperscript{27} We estimate the volatilities of the different forward rates. We are merely assuming that a given rate for a point in the future has the same volatility as that time point draws nearer. The empirical estimates we present here suggest that this is a reasonable assumption. Consider for example, the forward rate for a one-period bond six periods hence. For the first time step in the tree, this volatility is denoted as $\sigma(0,6)$. For the second time step, this rate is denoted as $\sigma(1,6)$. The remaining rates are $\sigma(2,6)$, $\sigma(3,6)$, $\sigma(4,6)$, $\sigma(5,6)$.\textsuperscript{28} Again, we assume that all of these volatilities are the same. This might not be a reasonable assumption if we find empirically that the volatilities of the forward rates are not relatively close.

Table 3 provides the estimates of the statistical parameters of the one-period continuously compounded forward rates for various periods ahead. We see that the volatilities of forward rates for various periods ahead are remarkably stable, ranging from 1.51\% to 1.68\%. Note also the small variation in means, medians, maxima, and minima. These results suggest considerable stability in the statistical parameters of short-term forward rates. Perhaps this result should not be surprising. We are fitting a binomial tree to only the first 12 months of the term structure. If we were working with a much longer term structure, there might be cause for alarm.

Thus, in fitting only the short end of the term structure, it seems reasonable that we conclude that while the volatilities of forward rates differ, the volatility of a given forward rate is reasonably stable across time. For the first round of tests, we shall use the volatility estimates in Table 3 for the appropriate rates in fitting our term structures. In a second round of tests, we shall vary the volatility over the years in the sample.

C. Estimation of the Drift

Of equal and perhaps greater importance is the drift. As is well-known in the term structure literature, the arbitrage-free condition, also known as the Local Expectations Hypothesis, is equivalent to a specific drift formula. For the HJM model, the drift formula is determined completely by the volatilities of the forward rates. To obtain the drift, we use the technique of Grant and Vora (1999), which is briefly outlined here.

Consider a given forward rate \( f(t,T) \), which is the forward rate observed at time \( t \) for a one-period bond to begin at time \( T \). In a binomial tree, the change in the forward rate is

\[
\Delta f(t,T) = \alpha(t,T)h + \sigma(t,T)\Delta W(t,T),
\]

\textsuperscript{27}Flesaker (1993) finds that the one-factor HJM model with constant volatility, i.e., the Ho-Lee model, performs poorly.
\textsuperscript{28}There is no $\sigma(6,6)$ because the one-period rate at time 6 is the spot rate, which has no volatility. It is just the rate on a zero coupon bond expiring in one period.
where \( h \) is the length of the time interval, \( \alpha(t,T) \) is the drift or expected change in the forward rate, \( \sigma(t,T) \) is the volatility of the forward rate, and \( \Delta W(t,T) \) is a discrete time random walk that takes on a value of \(-1\) or \(+1\) with equal probability. Without loss of generality, \( h \) can be standardized to a value of 1. Consistent with HJM and the requirement of no arbitrage, Grant and Vora show that the drift term, \( \alpha(t,T) \), is constrained by the volatility and is given by the following formula:

\[
\alpha(t,T) = \sigma(t,T) \sum_{j=t+1}^{T} \sigma(t,j) - \frac{1}{2} \sigma(t,j)^2.
\]

They then show that this term can be obtained from the covariance matrix of forward rate volatilities as one-half the sum of the \( T^{th} \) row and column. Alternatively, it can be calculated directly from the simple formula above.

D. Fitting the Tree and Estimating the Futures Price

For a set of forward rates of 1, 2, …, 11 periods ahead, we can obtain a tree of up to 11 periods. We need estimates of the prices of futures contracts on a three-month Eurodollar time deposit. Thus, the data available allow us to obtain these estimates for up to nine months to expiration. In other words, at time 9, we would have the current one-period spot rate, the one-period ahead one-period forward rate, and the two-period ahead one-period forward rate. With each period being one month, this information would be sufficient to price a three-month spot Eurodollar time deposit. The rate on this deposit would be the rate at which the futures contract expiring in nine months is priced at expiration. We can, thus, price futures contracts with 1, 2, …, 9 months to expiration where the underlying is a three-month Eurodollar.

To illustrate, for a given day, starting in January 1987, we fit a one-factor HJM model to the tree of LIBOR rates extending out for nine months. We then calculate the price at time 0 of a futures contract expiring in 1, 2, …, 9 months. The procedure is obtained by taking the spot price at expiration of a three-month Eurodollar. Suppose at an expiration point denoted as time \( T \), the one-month continuously compounded spot rate is 6\%, the one-month continuously compounded forward rate one-month ahead is 6.25\%, and the one-month continuously compounded forward rate two months ahead is 6.5\%. Thus, the price of a three-month Eurodollar time deposit paying \$1 at maturity is

\[
\exp[-.06(30/365)]\exp[-.0625(30/365)]\exp[-.065(30/365)] = 0.9847.
\]

This price implies a spot 90-day LIBOR of

\[
L(T,T+90) = \left( \frac{1}{0.9847} \right) - 1 \left( \frac{360}{90} \right) = 0.0622.
\]
Assuming the standard settlement procedure, the price of an expiring futures contract would then be

\[ F(T, T) = 1 - 0.0622 \left( \frac{90}{360} \right) = 0.9845. \]

Observe the difference between the forward price (0.9847) and the futures price (0.9845). Since this is an expiration price, this difference is unaffected by the marking-to-market process and is entirely attributable to the settlement procedure used to determine the futures price at expiration. As we noted in Section IB, this differential would be 1-10 basis points for a reasonable range of LIBORs.

To obtain the futures price prior to expiration, we start at expiration where the futures price is determined for all possible binomial outcomes. Then, stepping back in the tree, the futures price one period prior is the martingale probability-weighted average of the next two possible futures prices. We then repeat the procedure, stepping back through all outcomes and time steps until reaching time 0, at which point we have the futures price.

As noted above, the forward prices are easy to obtain directly from the spot prices. Hence, they are not influenced by the chosen term structure model or the volatility estimates. As noted, however, the standard forward prices are based on the assumption that the Eurodollar time deposit is an add-on instrument. If we compare them to the futures prices, wherein the underlying is treated as though it were a discount instrument, we induce some bias. We shall, therefore, have to address this concern.

V. Empirical Results of the Forward-Futures Price Differential

Using an Arbitrage-Free Binomial Tree

In this section we provide the results for estimates of the differences between forward and futures prices using the HJM model fit to daily term structure data over 1987-2000 under two conditions. In the first set of results we estimate the futures price under the assumption that the contract settles in the manner it does in practice, that is, as though the underlying is a discount instrument. We remove that bias in the second set of results by forcing the contract to settle at expiration as though the underlying is an add-on instrument, as it actually is. Although the first set of results is a better approximation of reality, the second set of results more accurately captures the effect of the daily marking-to-market that is the primary distinction between the cash flow streams of futures contracts and forward contracts.

\[ ^{29} \text{In most term structure models, the martingale probabilities are typically assumed to be 0.5, though this is not required. We do use 0.5 here.} \]
Let us also note that in all of the results that follow, we conducted t-tests for the significance of the mean differences. These differences are overwhelmingly significant, with most of the t-statistics greater than 10 and some more than 10,000. But these statistics are not reliable, because we do not know the sampling properties of these estimates. Our testing procedure is not equivalent to drawing a series of independent random samples. Therefore, the inferences that follow will be drawn less formally.

A. Differences in Forward and Estimated Futures Prices under the Assumption that Futures Contracts Settle Using the Discount Method

Table 4 shows that the average difference between forward and futures prices is 2.6 to 7 basis points with a standard deviation of about 1.2 basis points. The smallest difference is in futures contracts expiring in one month. The largest is in contracts expiring in nine months. Notice that the range is, however, relatively large. For contracts expiring in one month the maximum is about 10 times the minimum. For contracts with later expirations, however, the maximum is about twice the minimum. The differences across all expirations range from 0.7 to about 12 basis points. Nonetheless, these results do show smaller differences than reported by Meulbroek, are similar to those reported by Grinblatt and Jegadeesh, and slightly smaller than those we reported in Section III, all using empirical data. In addition, these results are free of any non-synchronicity problems or biases in empirical data due to market imperfections. Most importantly, none of these results are positive, as is typically observed in empirical data and would be completely inconsistent with rational investors.

It is worthwhile to note, however, that even though the mean difference is smallest for contracts expiring in one month, the standard deviation is the largest and the range is considerably larger than for contracts with later expirations. Ordinarily, pricing a futures contract expiring in the first time step of a binomial tree would lead to a futures price exactly equal to the forward price, because there is no interim marking-to-market. Yet in this case, not only are the two not equal, but the one-month contract has the largest standard deviation and a maximum difference ten times its minimum. More stable results are obtained for contracts maturing at later time steps. In fact, the most reliable results are obtained for contracts maturing the latest. These subtle findings suggest that something is awry. Indeed, the settlement feature of Eurodollar futures contracts is the culprit.

B. Differences in Forward and Estimated Futures Prices under the Assumption that Futures Contracts Settle Using the Add-on Method

To remove the bias from the fact that Eurodollar futures settle by the discount method, as explained above, we construct a hypothetical futures contract that settles by the add-on method
and is priced off of the term structure, the evolution of which is modeled by the Heath-Jarrow-Morton model. The results are presented in Table 5.

The first point to notice is the results for the futures contract expiring in one month, or equivalently, one binomial time step. As noted, a futures contract with a single time period would be equivalent to a forward contract, and, therefore, should have the same price. We do observe a very small difference, but it is not measurable until the 6th digit and amounts to about 0.08 basis points. The fact that it is not precisely zero can be easily explained. The Grant-Vora technique to estimate the drift is based on the assumption that a binomial tree converges to a continuous-time normal distribution in the limit. Clearly, this convergence cannot occur with one time step. Nonetheless, the difference is extremely small and for all intents and purposes is zero. Note also that the standard deviation is undetectable through seven digits, and there is barely a difference between the extremely small maximum and minimum. These results for a one-month futures contract are something of a litmus test for the reliability of this estimation procedure, and it appears as if it passes the test.

For the remaining contracts, the mean differences range from –0.26 basis points to –4.1 basis points, with the magnitudes of the differences increasing with contract maturity. The mean differences in Table 4 are from about 2 to over 33 times the mean differences in Table 5, meaning that the effect of the futures contract settling as a discount instrument has a significant impact on the estimated forward-futures differential.

The standard deviations in Table 4 are from over 80 to over 4,000 times the corresponding ones in Table 5, where the largest standard deviation is but 0.015 basis points. We remarked on the ranges in Table 4. In Table 5, the differences between maximum and minimum values are extremely small for all maturities. These results are remarkably stable, especially considering they are estimated from over 3,500 term structures over 14 years. Thus, while the magnitudes of the differences between futures and forward prices attributed to marking-to-market are smaller once the settlement feature is eliminated, the reliability of the estimated differences is also considerably greater.

C. Improved Estimates Using Differential Volatilities Across Time

As noted earlier, we used volatility estimates that were based on an overall volatility for the time period. We obtained a volatility for the one-month ahead forward rate, the two-month ahead forward rate, on up to the nine-month ahead forward rate. The forward rate volatilities do vary from year to year, as indicated in Figure 2. We see that forward rate volatilities were

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30It is tempting to suggest that the large sample size leads to stability of these estimates, but the same term structures are used to estimate the results in Table 4, which are much less reliable.
relatively high in certain years and much lower in others. Of course, the differential volatility across forward rate maturities is taken care of by our using different volatilities for different forward rate maturities. But the differences in volatilities across time have not been considered.

Accordingly, the tests are rerun in two ways. We run the tests for the overall period (1987-2000) but use a different set of forward rate volatilities each year. These volatilities are estimated from the data for that specific year. For example, when we run the tests using data from a given year, we use forward rate volatilities estimated using data from that same year. Then we also run the tests separately for each year in the overall period, using, of course, the appropriate forward rate volatilities for the respective years. These tests are done using the hypothetical futures contract, which settles based on the add-on method. The results are presented in Table 6. Due to the large amount of information, we show only the average differences.

The column labeled “Overall” gives the results for the full time period, using the appropriate volatility for each year within that period. The largest average is about 0.7 basis points. The mean differences in Table 5, obtained without varying the volatility by year, are anywhere from 5.8 to 9.6 times the mean differences for the overall period in Table 6, where the volatility is varied appropriately by year.\(^{31}\) The mean differences always increase as the maturity increases, a result we showed in both previous tables.

Another interesting finding is that if the average differences for each of the 14 years are sorted in ascending order, the ordering corresponds identically to that if the average of the estimated forward rate volatilities for each of the 14 years is also sorted in ascending order. In other words, without fail, if interest rates are more volatile in a particular year, the estimated differences between forward and futures prices is greater. This point was also observed by Grinblatt and Jegadeesh in their data.

The remaining columns in Table 6 show the results year-by-year with the appropriate volatility used for the respective year. The largest difference over the entire table is 1.9 basis points. In 10 of the 14 years, the largest difference is less than one basis point.

\[D. \quad \textit{The Effect of Estimating Volatilities from Historical Data}\]

\(^{31}\)Although not shown, the results when the volatilities are varied by year (Table 6) have higher standard deviations than in the case where the volatilities are not varied by year (Table 5). This result should not surprising, however, since the altering of volatility from year to year should definitely result in less stability in the estimates. The volatilities are still considerably smaller than those reported in Table 4 where the settlement feature corrupts the results. Tests in which the futures settles by the add-on method but the volatility varies by year resulted in standard deviations that are over 2.8 to 165 times those obtained when the settlement feature is not a factor and the volatility is varied year by year. So, when the volatility is varied by year, the standard deviations will be higher than when a single volatility is used for each forward rate over the full time period, regardless of how the settlement feature is accounted for.
Although the Heath-Jarrow-Morton model assumes a deterministic volatility structure, as in most financial models, the parameters must be estimated, which nearly always requires the use of historical data. In this section we examine the effect of estimating the volatilities from historical data.

First, let us recall that the results of this study are based on one source of input information, the set of spot prices, derived from the term structure of spot rates, of Eurodollar time deposits. From this information, we estimated the forward rates and from these forward rates, we estimated their volatilities. As shown in the previous section, the use of more contemporaneous estimates of volatilities leads to much smaller estimates of the difference between forward and futures prices. This volatility information would not, however, be known to traders at the time at which they are computing arbitrage-free prices. Traders would have had to forecast the volatilities.

There are an infinite number of possible forecasting models. The objective of this paper is not to address the myriad of complex issues associated with interest rate volatility forecasting.\(^{32}\) Thus, we examine the effect of a very simple forecasting model. We take the series of daily forward rates for one year, estimate the volatility of each forward rate, use that estimate to fit the current term structure, and conduct the tests using the following year’s data. This procedure is updated each year. Given the data set that covers 1987-2000, the tests span the period 1988-2000. We conduct these tests using the assumption that the futures contract is constructed as an add-on instrument.

A representative sample of the results is shown in Table 7 where we show the findings for the entire time period and for the years 1988, 1992, 1996 and 2000 for futures contracts expiring in one month, three months, six months, and nine months. The omitted results follow the same patterns. For each combination of contract maturity and time period, we show two numbers. The top number is the difference between the futures and forward prices using the volatility forecasted for the given time period. The number below and in parentheses is the number, taken from Table 6, representing the average difference between the futures and forward prices where the volatility is estimated contemporaneously. For example, for a six-month contract in 1992, the average difference obtained when fitting 1992 term structure data using volatilities estimated with 1991 data is –0.476 basis points. The average difference obtained when fitting 1991 term structure data using volatilities estimated with 1991 data is –0.473 basis points.

\(^{32}\)See, for example, Brenner, Harjes, and Kroner (1996) and Chan, Karolyi, Longstaff, and Sanders (1992).
First observe the results for the overall time period. When forecasting the volatilities, the differences are still negative but slightly more negative. Nonetheless, the differences are on a very similar order of magnitude to those obtained with contemporaneous volatility estimates. Thus, even with a very crude forecasting model, we obtain virtually the same results as if we had perfect foresight of contemporaneous volatility. \(^{33}\) For each individual year examined, observe that the average differences are extremely close to the average differences of the previous year when contemporaneous volatility estimates were used. These findings are consistent throughout Table 7 and also in the results not reported in the table. This finding implies that the volatility of forward rates and not the level of the term structure is the primary driver in determining the difference between futures and forward prices. Yet, even when volatility must be forecasted and only an extremely simple model is used, the results are almost indistinguishable from those that use contemporaneous volatility, which would require perfect foresight.

VI. Summary and Conclusions

There has been very little research on the differential between forward and futures prices and even less for the most active forward and futures market, the Eurodollar/LIBOR market. One reason is that obtaining sufficient data has been a difficult problem. A more subtle issue, however, is the different pricing procedures used in the Eurodollar spot and futures markets. It is not a simple matter to adjust for this pricing bias. This paper shows that the use of actual empirical data can be misleading, providing the logical result that the forward price often exceeds the futures price on average but that in far too many cases the futures price exceeds the forward price. Moreover, there appears to be no way to properly adjust the empirical data for this settlement bias. As we show here, a sensible solution to the problem is to fit an arbitrage-free term structure model to the forward rate data and price a futures contracts not as it is actually priced, as a discount instrument, but as an add-on instrument, consistent with the underlying.

In comparing these results to those of other researchers, let us recall what others had found. Meulbroek found average differences of –2.2, -23.7, and –47.8 basis points for different maturity ranges. Grinblatt and Jegadeesh found average differences of -3 to -5 basis points but ranges of –0.5 to -10.2 basis points. Recall too that Grinblatt and Jegadeesh found a number of

\(^{33}\)The only other factor that could have an impact on these comparisons is the fact that the tests using forecasted volatilities cover the 1988-2000 period and the tests using the contemporaneous volatilities cover the additional year of 1987. The tests using contemporaneous volatilities for the overall period but excluding 1987 resulted in average differences for one, three, six, and nine months of –0.007, -0.062, -0.264, and –0.656 basis points. Thus, removing the 1987 results with contemporaneous volatilities results in similar but slightly smaller differences. Hence, these results with forecasted volatilities are slightly more negative when 1987 is removed from the benchmark, but they are still on a very similar order of magnitude.
cases where rate differences were negative, leading to positive price differences. Of course, if investors are rational, these positive price differences should never occur with Eurodollar instruments and bias the negative values toward zero, suggesting that the differences between forward and futures prices are smaller than they actually are. Sundaresan, using arbitrary inputs into a formula, obtained differences of –2 and –6 basis points, depending on maturity. Using a very small sample of empirical data, he obtained an average difference of –12.5 basis points. Discussions with the CME indicate that a typical bid-ask spread is about one-half basis point for contracts of up to two years expiration. Thus, all of the previous findings suggest differences that exceed the bid-ask spread by a considerable margin.

The results of the present study show that the differential is probably much smaller. Fully taking into account the unusual settlement feature of Eurodollar futures and fitting a daily data set of LIBOR term structures over a 14 year period to the Heath-Jarrow-Morton model, it provides estimates that are smaller and appear to have considerably less variation. The present study also shows that even smaller differences can be obtained if volatility estimates that are more contemporaneous are used. Seven of the 14 largest annual average differences using contemporaneous estimates of the volatility are less than the bid-ask spread, and most of the remaining average differences do not exceed the typical bid-ask spread by much. The overall average difference of about 0.7 basis points is only slightly higher than the typical bid-ask spread of 0.5 basis points. Moreover, we show that volatility can be forecasted sufficiently well so that these results are supported with only a simple forecasting model.

The central question is whether forward and futures prices are equal for futures contracts in general. Of course there are many reasons why they might not be equal: default risk, taxes, liquidity, transaction costs, etc. But the primary concern of most of the literature is whether the daily marking-to-market procedure, which leads to different cash flow streams, results in differences in forward and futures prices. To address this question, these other potential factors must be eliminated. The extreme depth of the Eurodollar market makes it, on the one hand, an excellent market for testing this question. On the other hand, the unusual settlement feature of Eurodollar futures is a complication. We show that it is imperative to recognize this feature and eliminate it, along with other complications, to draw the most reliable conclusions.

Granted that we cannot truly fit an HJM tree divided into daily units, nonetheless, HJM trees of the sort used here are commonly employed in practice. There is considerable belief that trees of this sort provide reliable prices for futures contracts. Trees fit with daily time steps are not possible, given the requirement of a term structure with maturities spaced one day apart. Dealers commonly use the Eurodollar futures contract to hedge their positions in LIBOR-based
swaps, forward contracts, and interest rate options. Futures contracts would not successfully provide a hedge for these positions if they were not priced relatively well by term structure models such as this.

We have found that the differences between futures and forward prices in the Eurodollar market are consistently negative, as they should be, but essentially no different from zero. It seems reasonable to infer that in markets in which the linkage between futures prices and interest rates is weaker and less direct, the differences attributable to marking-to-market would be even closer to zero.
REFERENCES


Burghardt, Galen and Bill Hoskins, 1995a, The Convexity Bias in Eurodollar Futures: Part 1, *Derivatives Quarterly* 1, Spring, 47-55.


Table 1. Statistical Properties of the Difference Between Eurodollar Futures Prices and Forward Prices

Spot LIBOR rates are obtained from the British Bankers Association web site (www.bba.org), which consists of rates for each business day for Eurodollar time deposits of 1 to 12 month maturities, covering the period 1987-2000. Forward prices are then inferred from the implied forward rates. Futures prices are for the 90-day Eurodollar contract of the Chicago Mercantile Exchange, which are obtained from the Institute for Financial Markets and cover the same period. The expiration dates of the futures contracts are matched with the expiration dates of forward contracts and the difference, calculated as “futures price – forward price”, is obtained. Means, standard deviations, maximums, and minimums are in basis points.

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<tr>
<th>Expiration (months)</th>
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<th>Minimum</th>
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Table 2. Statistical Properties of the Difference Between Eurodollar Futures Prices and Adjusted Forward Prices

Spot LIBOR rates are obtained from the British Bankers Association web site (www.bba.org), which consists of rates for each business day for Eurodollar time deposits of 1 to 12 month maturities, covering the period 1987-2000. Forward prices are then inferred from the implied forward rates. Futures prices are for the 90-day Eurodollar contract of the Chicago Mercantile Exchange, which are are obtained from the Institute for Financial Markets and cover the same period. The expiration dates of the futures contracts are matched with the expiration dates of forward contracts and the difference, calculated as “futures price – forward price”, is obtained. The adjusted forward price is based on the assumption that Eurodollar time deposits are priced as discount instruments, whereby the interest is deducted from the face value in advance, in contrast to the standard procedure in which the interest is added on to the principal. Means, standard deviations, maximums, and minimums are in basis points.

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Spot LIBOR rates are obtained from the British Bankers Association web site (www.bba.org), which consists of rates for each business day for Eurodollar time deposits of 1 to 12 month maturities, covering the period 1987-2000. Forward rates are computed from spot rates and converted to continuous compounding. The number of observations is slightly more than 3,500.

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<td>f(0,11)</td>
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Spot LIBOR rates are obtained from the British Bankers Association web site (www.bba.org), which consists of rates for each business day for Eurodollar time deposits of 1 to 12 month maturities, covering the period 1987-2000. Forward prices are then inferred from the implied forward rates. A one-factor Heath-Jarrow-Morton is fit to the term structure of LIBOR spot rates using a binomial tree. Volatilities are estimated from forward rate data over the entire time period. The tree is fit using nine time steps, each representing one month. The futures prices is obtained by starting at the expiration with the three-month spot LIBOR, applying the equivalent martingale principal and working backwards through the tree to time 0. The number of observations is slightly more than 3,500. Means, standard deviations, maximums, and minimums are in basis points.

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<table>
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Table 6. Average Basis Point Differences Between Futures and Forward Prices Using Different Volatilities Overall and by Year

Spot LIBOR rates are obtained from the British Bankers Association web site (www.bba.org), which consists of rates for each business day for Eurodollar time deposits of 1 to 12 month maturities, covering the period 1987-2000. Forward prices are then inferred from the implied forward rates. A one-factor Heath-Jarrow-Morton is fit to the term structure of LIBOR spot rates using a binomial tree. Volatilities are estimated from forward rate data separately for each year. The tree is fit using nine time steps, each representing one month. The futures prices is obtained by starting at the expiration with the three-month spot LIBOR, assuming that the futures price is based on the add-on method of interest, applying the equivalent martingale principal to work backwards through the tree to time 0. Thus, the futures price is arbitrage-free. The results labeled “Overall” cover the entire time period of 1987-2000 but calculations for observations of a given year use forward rate volatilities that are estimated using data for that year. The number of observations is slightly more than 3,500.

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Table 7. Average Basis Point Differences Between Futures and Adjusted Forward Prices Using Different Volatilities Overall and for Selected Years

Spot LIBOR rates are obtained from the British Bankers Association web site (www.bba.org), which consists of rates for each business day for Eurodollar time deposits of 1 to 12 month maturities, covering the period 1987-2000. Forward prices are then inferred from the implied forward rates. A one-factor Heath-Jarrow-Morton is fit to the term structure of LIBOR spot rates using a binomial tree. Volatilities are estimated from forward rate data separately for each year. The data from a given year are used to estimate the volatilities of each forward rate, which are then used as volatilities for the following year. The tree is fit using nine time steps, each representing one month. The futures prices is obtained by starting at the expiration with the three-month spot LIBOR, assuming that the futures price is based on the add-on method of interest, applying the equivalent martingale principal to work backwards through the tree to time 0. Thus, the futures price is arbitrage-free. The results labeled “Overall” cover the entire time period of 1988-2000 but calculations for observations of a given year use forward rate volatilities estimated using data for the previous year. The overall number of observations is about 3,300. The number in the top of the cell is the average difference in the futures and forward price. The number in the bottom of the cell is the average difference in the futures and forward price from the previous year (see Table 6), where the volatility is estimated contemporaneously.

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Figure 1. The Difference Between the Forward and Futures Price per $1 Par at Expiration for a Range of LIBOR

The forward price is computed as $1/(1 + \text{LIBOR}(90/360))$, and the futures price is computed as $1 – \text{LIBOR}(90/360)$.
Figure 2. Year-by-Year Volatility Estimates

Spot LIBOR rates are obtained from the British Bankers Association website (www.bba.org.uk), which consists of rates for each business day for Eurodollar time deposits of 1 to 12 month maturities, covering the period 1987-2000. Forward rates are computed from spot rates and converted to continuous compounding. The number of observations is slightly more than 3,500.
Appendix

Proposition 1: The expiration settlement feature of the Eurodollar futures contract is sufficient but not necessary to prevent a perfect hedge.

First note that a perfect hedge between a spot transaction and a forward or futures contract linked to the same source of risk is sufficient to establish the cost of carry model for pricing the forward or futures contract. It is well-known that a forward contract provides a perfect hedge. For a futures contract, we examine the case of an anticipatory loan hedged with a futures.

Proof: Consider a borrower at time 0 who identifies the need to borrow $1 at time t. The borrower constructs a hedge using a futures contract. The futures contract is not marked-to-market until expiration. LIBOR at time t is denoted $L_t$. The futures price at expiration is $f_t(t) = 1 - L_t$.

\[ \text{Time 0} \]
Sell one futures contract that expires at t at price $f_0(t)$

\[ \text{Time t} \]
The futures payoff is $-(f_t(t) - f_0(t)) = -(1 - L_t - f_0(t))$

To generate $1, borrow B such that

\[-(1 - L_t - f_0(t)) + B = 1\]

Thus,

\[B = 1 - L_t + 1 - f_0(t)\]

\[ \text{Time T} \]
If the loan were an add-on note, pay back the amount

\[(1 - L_t + 1 - f_0(t))(1 + L_t) = 1 - f_0(t) + L_t - f_0(t)L_t + L_t^2\]

The existence of terms related to LIBOR establishes that this is not a perfect hedge. This proves sufficiency. If the loan were issued as a discount note, pay back the amount

\[(1 - L_t + 1 - f_0(t))/(1 - L_t) = ((1 - f_0(t))/(1 - L_t) + 1\]

This result contains a LIBOR term and, therefore, proves that the expiration settlement feature is not necessary to prevent a perfect hedge.
Proposition 2: Had the Eurodollar futures contract been designed as an add-on instrument, a perfect hedge would be possible.

Proof: Consider the case of issuing an add-on note. The company should sell \(1/f_0(t)\) futures. The futures payoff would now be \(-(1/f_0(t))(1/(1+L_t) - f_0(t))\). The amount borrowed would be the solution

\[ -(1/f_0(t))(1/(1+L_t) - f_0(t)) + B = 1 \]

so that

\[ B = 1/((1 + L_t)f_0(t)) \]

The amount repaid at \(T\) on an add-on note would be

\[(1/((1 + L_t)f_0(t)))(1 + L_t) = 1/f_0(t)\]

This amount is known at time 0. Thus, the hedge is perfect, and the futures contract would be priced the same as the forward contract. It is easy to demonstrate that a perfect hedge is not possible if the loan were a discount note.

Propositions 1 and 2 demonstrate that the expiration settlement feature of the Eurodollar futures contract is a sufficient condition to produce a forward-futures price differential that is distinct from other sources of the forward-futures differential such as the timing difference and marking-to-market.