Options on foreign currencies can be somewhat more difficult to understand than options on stocks, bonds, and commodities. The reason primarily lies in the fact that every currency contract involves not one but two currencies. Options on stocks, bonds, and commodities, by contrast do not involve more than one currency. They do implicitly involve more than one asset. For example, an option to buy a stock is an option to exchange a currency for the stock. From the opposite perspective, however, it is clearly an option to exchange stock for the currency. We may speak of a share of stock as costing $50 but we could just as easily say that the cost of $1 is 0.02 shares of stock. That is, one could theoretically exchange 0.02 shares of stock and receive $1. Although the notion of exchanging stock for money is a very valid economic concept and explains precisely the same thing as if we described the transaction as exchanging money for stock, as we usually do.

Hence, exchanging dollars for euros is the mirror image of exchanging euros for dollars. Likewise, exchanging dollars for stock is the mirror image of exchanging stock for dollars. Since we never describe the stock transaction as the latter, we typically have trouble understanding the transaction from that point of view.

In the world of currency trading, however, it is necessary to understand the transaction from either point of view. In particular, foreign currency option transactions have an important characteristic that sheds light on some interesting issues in option valuation. That characteristic is that a foreign currency call can be viewed as a put from the mirror image perspective. We describe that as the concept of isomorphism, the notion that two things are essentially the same. This point would also hold for calls on stocks since they can be viewed as puts on cash, but because we virtually never view a stock purchase as the purchase of money paid for by tendering stocks, we never use this concept elsewhere in the financial markets.
In foreign currency transactions, the two currencies are described as the base currency and the quoted currency. The base currency is just the underlying currency. For example, a call option quoted in dollars in which the underlying is euros has the dollar as the quoted currency and the euro as the base currency.

**Definition of Terms**

Define the following terms:

- $S_{0,T}$ = spot exchange rate at times 0, $T$, quoted in the domestic currency, i.e., units of the domestic currency per unit of foreign currency, the base currency
- $R_{0,T} = 1/S_{0,T}$, the spot exchange rate at times 0, $T$, quoted in the foreign currency, i.e., units of the foreign currency per unit of domestic currency, the base currency
- $X$ = exercise price quoted in the domestic currency of a call option to buy one unit of the foreign currency
- $T$ = time to expiration of the option
- $\sigma$ = volatility of the exchange rate
- $r_D$ = domestic risk-free interest rate
- $r_F$ = foreign risk-free interest rate

**Option Payoffs**

The payoff of the call at expiration is

\[
S_T - X \quad \text{if} \quad S_T > X \\
0 \quad \text{if} \quad S_T \leq X
\]

We now show that this option is isomorphic to a put option denominated in the foreign currency to sell $X$ units of the domestic currency at an exercise price of $1/X$. Let $X_R = 1/X$, the reciprocal strike. First we note that the exercise condition of the put can be expressed as

In-the-money: $1/S_T \leq 1/X \equiv S_T > X$

Out-of-the-money: $1/S_T > 1/X \equiv S_T \leq X$

Thus, the put in-the-money condition is the same as the call in-the-money condition. The payoff of a single put option is

\[
X_R - R_T = (1/X) - (1/S_T) \quad \text{if} \quad S_T > X \\
0 \quad \text{if} \quad S_T \leq X
\]
Since there are X put options, this payoff in aggregate is
\[ X[(1/X) - (1/S_T)] = 1 - (X/S_T) \quad \text{if} \quad S_T > X \]
0 \quad \text{if} \quad S_T \leq X
This payoff is in units of the foreign currency. To convert to domestic currency, we multiply by \( S_T \).
\[ S_T(1 - (X/S_T)) = S_T - X \quad \text{if} \quad S_T > X \]
0 \quad \text{if} \quad S_T \leq X
Thus, we see that the payoff of the put is the same as the call.

**Valuation of the Options**

Now we demonstrate how the premiums of these two options are equivalent in the Black-Scholes-Merton model. The BSM value of the call option is
\[
c = S_0 e^{-r_0 T} N(d_1^D) - X e^{-r_0 T} N(d_2^D)
\]
where
\[
d_1^D = \frac{\ln(S_0 e^{-r_0 T} / X e^{-r_0 T}) + (\sigma^2 / 2)T}{\sqrt{T}}
\]
\[
d_2^D = \frac{\ln(S_0 e^{-r_0 T} / X e^{-r_0 T}) - (\sigma^2 / 2)T}{\sqrt{T}}
\]
Note that we use the “\( D \)” superscript on the normal distribution function to indicate that this is the normal probability for the call written on the foreign currency and denominated in the domestic currency. As we stated, this call is equivalent to \( X \) puts written on the domestic currency denominated in the foreign currency with strike of \( X_R = 1/X \). We now validate that this result is upheld in the BSM model.

From the foreign investor’s perspective, the value of a single unit of the isomorphic put
\[
p = X_R e^{-r_0 T} N(-d_2^F) - R_0 e^{-r_0 T} N(-d_1^F)
\]
where \( p \) is denominated in the foreign currency and\(^1\)
\[
d_1^F = \frac{\ln(R_0 e^{-r_0 T} / X_R e^{-r_0 T}) + (\sigma^2 / 2)T}{\sqrt{T}}
\]
\[
d_2^F = \frac{\ln(R_0 e^{-r_0 T} / (X_R e^{-r_0 T}) - (\sigma^2 / 2)T}{\sqrt{T}}
\]
---

\(^1\)Keep in mind that the reference to an interest rate being “\( D \)” for domestic or “\( F \)” for foreign must maintain consistency. Even though we are viewing this transaction from the foreign investor’s point of view, the rate denoted “\( F \)” is his rate and “\( D \)” is the foreign rate from his point of view.
Let examine why this is the correct put formula. In words, the BSM formula for the value of a put is the strike times the present value factor for the investor’s currency times $N(-d_2^F)$ minus the underlying price in the investor’s currency times the present value factor for the yield on the currency times $N(-d_1^F)$. Thus, in the above formula for $p$, the strike is properly designated as $X_R = 1/X$. The strike is then multiplied by the present value factor for the investor’s currency. Since this put is being valued from the foreign investor’s perspective, it is appropriate to discount the strike at the foreign rate. Then we have the normal probability based on $-d_2^F$, with the $F$ superscript to distinguish it from $d_2^D$ in the call. From the product of these three terms, we subtract the exchange rate in foreign currency, $R_0 = 1/S_0$, times the present value factor for the opposite currency rate. From the foreign investor’s perspective, this rate is $r_D$. Then this product is multiplied by the normal probability for $-d_1^F$.

Now let us work with $d_1^F$ and $d_2^F$. First, note the following

$$\ln(R_0 e^{\gamma_0 T} / X e^{\gamma_T}) = \ln((1/S_0) e^{-\sigma^2 / 2}T / (1 - X) e^{-\gamma_0 T})$$

$$= \ln(1) - \ln S_0 - r_D T - \ln 1 + \ln X + r_T T$$

$$= -\ln S_0 - r_D T + \ln X + r_T T$$

$$= -\ln(S e^{-\gamma_0 T} / X e^{-\gamma_T})$$

Thus,

$$d_1^F = \frac{-\ln(S e^{-\gamma_0 T} / X e^{-\gamma_T}) + (\sigma^2 / 2)T}{\sigma \sqrt{T}}$$

$$= -\left(\frac{\ln(S e^{-\gamma_0 T} / X e^{-\gamma_T}) - (\sigma^2 / 2)T}{\sigma \sqrt{T}}\right)$$

$$= -d_2^D$$

$$d_2^F = \frac{-\ln(S e^{-\gamma_0 T} / X e^{-\gamma_T}) - (\sigma^2 / 2)T}{\sigma \sqrt{T}}$$

$$= -\left(\frac{\ln(S e^{-\gamma_0 T} / X e^{-\gamma_T}) + (\sigma^2 / 2)T}{\sigma \sqrt{T}}\right)$$

$$= -d_1^D$$

Therefore,

$$N(-d_1^F) = N(d_2^D)$$

$$N(-d_2^F) = N(d_1^D)$$

Our put formula will, thus, be
Recall that the isomorphic put is actually $X$ puts, so the aggregate put premium is

$$Xp = X(1/X)e^{-rT}N(d_1^D) - X(1/S_0)e^{-rT}N(d_2^D)$$

This value is in units of the foreign currency. To convert to the domestic currency, we simply multiply by $S_0$.

$$(Xp)^D = S_0e^{-rT}N(d_1^D) - Xe^{-rT}N(d_2^D)$$

This is the same as the right-hand side of the call.

**Probability of Exercise**

It is well-known that the risk-neutral probability of exercise of the call option is $N(d_2^D)$. This is the risk-neutral probability that $S_T > X$. The corresponding probability for exercise of the isomorphic put is the probability that $1/S_T < 1/X$, which has to be the same as the probability that $S_T > X$. Thus, $N(d_2^D)$ is the risk-neutral probability of exercise of the call, and this equals $N(-d_2^F)$, which is the probability of exercising the put. Note that the probability of exercising the put is not the term in the BSM put formula that is multiplied by the present value of $1/X$, or $N(-d_2^F)$. There is an important reason for that, which will become clearer when we examine the actual, as opposed to risk-neutral, probability of exercise.

Let the true stochastic process for $S$ be given as

$$dS_t/S_t = \alpha dt + \sigma dW_t$$

Here $\alpha$ and $\sigma$ are the expected return and volatility of $dS_t/S_t$. It is important to remember that the stochastic process for the log return is

$$d\ln S_t = \mu dt + \sigma dW_t$$

with the log expected return given as

$$\mu = \alpha - \sigma^2/2$$

which holds by definition for normal distributions in continuous time.

Now we need the stochastic process of the isomorphic put. We will apply Itô’s Lemma to $R_t = 1/S_t$, so we will need certain partial derivatives:

$$\frac{\partial R_t}{\partial S_t} = -\frac{1}{S_t^2}, \quad \frac{\partial^2 R_t}{\partial S_t^2} = \frac{2}{S_t^3}, \quad \frac{\partial R_t}{\partial t} = 0$$
Now, applying Itô’s Lemma to \( R_t \),

\[
dR_t = \frac{\partial R_t}{\partial S_t} dS_t + \frac{1}{2} \frac{\partial^2 R_t}{\partial S_t^2} dS_t^2 + \frac{\partial R_t}{\partial t} dt
\]

Substituting the above results and as well as \( dS_t/S_t = \alpha dt + \sigma dW_t \) and \( dS_t^2 = S_t^2 \sigma^2 dt \), we obtain

\[
dR_t = \left( \frac{1}{S_t} \right) (-\alpha dt - \sigma dW_t + \sigma^2 dt)
\]

\[
= \left( \frac{1}{S_t} \right) (-\alpha + \sigma^2) dt - \sigma dW_t
\]

Dividing the LHS by \( R_t \) and the equivalent operation on the RHS, multiplying by \( S_t \), we obtain:

\[
\frac{dR_t}{R_t} = (-\alpha + \sigma^2) dt - \sigma dW_t
\]

Thus, the expected return on the reciprocal currency is \(-\alpha + \sigma^2\). Note, however, the minus term on the volatility, which contributes to the reason why the probability of exercise of the put is not given by the \( N(-d_2) \) term in its formula, as it usually is in the BSM model.²

The log return on \( R_t \) is

\[
-\alpha + \sigma^2 - \sigma^2 / 2 = -\alpha + \sigma^2 / 2
\]

Substituting for \( \alpha \), this can then be expressed as

\[
-\alpha + \sigma^2 / 2 = -\left( \mu - \sigma^2 / 2 \right) + \sigma^2 / 2 = -\mu
\]

This is the term that must be inserted into the formula for the normal probability.³ It is the expected log return on \( R_t \) expressed in terms of the log return on \( S_t \). Making the substitution into \(-d_1^F\), and using a * to denote that we are dealing with the true probability distribution, not the risk-neutral distribution, we have

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²The probability in the BSM formula is based on a stochastic process in which the noise term, \( \sigma dW_t \), is added, not subtracted.

³This result is easy to see in the risk neutral context. The risk-free rate term in the numerator of \( d_1^D \) is \( r_D - r_f \) where under risk neutrality, \( r_D \) is the risk neutralized expected return on the domestic currency and \( r_f \) is the risk neutralized expected return on the foreign currency. From the foreign investor’s point of view, this corresponding term would be \( r_F - r_D \) which is \(-(r_D - r_f)\). That is, the expected return is converted to its negative value.
\[ N(-d_{1}^{*}) = N\left\{ \frac{\ln\left(\frac{R_{0}}{X_{R}}\right) + (-\mu + \sigma^{2}/2)T}{\sigma\sqrt{T}} \right\} \]
\[ = N\left\{ \frac{\ln(X/S_{0}) + (-\mu + \sigma^{2}/2)T}{\sigma\sqrt{T}} \right\} \]
\[ = N\left\{ \frac{-\ln(X/S_{0}) + (\mu - \sigma^{2}/2)T}{\sigma\sqrt{T}} \right\} \]
\[ = N\left\{ \frac{-\ln(X/S_{0}) + (\mu - \sigma^{2}/2)T}{\sigma\sqrt{T}} \right\} \]
\[ = N\left\{ \frac{\ln(S_{0}/X) + (\mu - \sigma^{2}/2)T}{\sigma\sqrt{T}} \right\} \]

This is, of course, \( N(d_{2}^{*}) \) since
\[ d_{2}^{*} = \frac{\ln(S_{0}/X) + (\mu - \sigma^{2}/2)T}{\sigma\sqrt{T}} \]

Some Terminology Confusion

Some foreign currency option traders do not like the terminology “put” and “call.” They see each option as being both a put and a call. Hence, it is common in foreign currency option transactions to use slightly different terminology than in other types of options. Foreign currency options, thus, are written with two values referred to as the “Call Currency Amount” and the “Put Currency Amount.” As with any option, the holder of the option makes the exercise decision. If he chooses to exercise, he tenders the Put Currency Amount and receives the Call Currency Amount.4 As example, suppose an option written on the pound designates the call currency amount as £10,000,000 and the put currency amount as $15,000,000. At expiration, if the exchange rate is higher than $1.50, the holder of the option tenders (“puts”) $15,000,000 to the writer and claims (“calls”) £10,000,000. With the exchange rate higher than $1.50, the option holder can exchange the £10,000,000 and receive more than $15,000,000.

In addition, there is a further matter in the world of currencies that causes some confusion. This involves the use of ratios in expressing exchange rates. I have some disagreement with the way the industry uses the concept. It can be quite confusing.

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4In this context, it is easy to see why options are referred to as puts and calls. The holder of the option “puts” one currency to the writer of the option and “calls” the other currency. In the context of assets other than currencies, the holder of the option “puts” the asset and calls the agreed-upon cash, the exercise price, or the other way around.
Consider an ordinary ratio, say 2/3. In elementary math, this ratio is easily interpreted. We mentally consider an object divided into thirds in which we reference two of those thirds. For example, we might say that a stadium is two-thirds full. Ratios can be easily multiplied. For example, (2/3) * (1/5) = 2/15. When we express the price of milk, we frequently say something like $3.00 a gallon. This can be written as $3/gal. where the “/” is interpreted as “per.” We say that $3 is the price per gallon of milk.

In the currency world, we can do a similar operation. We can indicate the ratio USD/EUR. In conformance with the notion that this is a ratio, “/” is interpreted as “per” and we read that ratio as “the number of dollars per euro.” Thus, if USD/EUR = $1.30, we say that one euro costs $1.30.

As it turns out, however, the currency world does just the opposite. The notation USD/EUR means “the number of euros per U. S. dollar.” The numerator is the base currency and the denominator is the quote currency. The multiplication words just the same and is just as valid. For example, using the dollar (USD), euro (EUR), and Japanese yen (JPY), we have (USD/EUR) * (EUR/JPY) = USD/JPY. This is read as euros per dollar times yen per euro equals yen per dollar, which is USD/JPY in the industry’s notation. This notational form is standard in the industry, but does appear reversed. If it were not, we could write the price of milk as gal./$3, which we would never do. Thus, you should be aware of this format.

My personal preference is to never write exchange rates in this manner. Although I cannot swear I have never done so, my books and articles typically write the exchange rate in a very clear form, such as the “the price or exchange rate of a euro is $3.”