Pricing a swap means to determine the terms of the swap at the start of the transaction. In the case of a plain vanilla (fixed-for-floating) interest rate swap, pricing means to determine the fixed rate that will be exchanged for the floating rate. The fixed rate is determined as the rate that establishes the present value of the fixed payments as equal to the present value of the floating payments. In the case of a currency swap, in which either or both sides can pay fixed or floating, pricing the swap establishes the notional principals in the two currencies. In all cases, pricing a swap is accomplished by identifying the terms that make the value of the swap zero to both parties at the start.\footnote{It is possible for a swap to be established at terms that do not set the market value at zero at the start. The parties would agree to an arbitrary fixed rate for an interest-rate swap or arbitrary notional principals for a currency swap. This type of swap is called an off-market swap. In such a case, the value of the swap to one of the parties is positive, so that party pays that amount to the other at the start. This type of arrangement is very much the exception and will not be considered in this note. We will, however, encounter an off-market instrument later.}

This teaching note uses an example to illustrate how interest rate and currency swap pricing is accomplished. We assume that the term structure of spot interest rates is given, and we let the notional principal for the dollar interest rate swap be $1.\footnote{Swap payoffs are linearly homogeneous with respect to the notional principal so the results for any notional principal can be obtained from these results by multiplying by the appropriate notional principal.} There is no need for a complete pricing model, such as Cox-Ingersoll-Ross or Heath-Jarrow-Morton, because swap pricing is somewhat analogous to futures pricing, wherein the payoffs are linear. Such models are necessary in pricing interest rate options, because option payoffs are nonlinear.

Another procedure related to swap pricing is valuation. Normally one thinks of pricing and valuation as the same process, at least in an efficient market. For swaps and forward contracts, however, pricing means to determine the terms of the transaction at the start. Valuation means to determine the market value of the transaction at a time later during its life. Pricing is, therefore, simply valuation at the start of the contract when the
value is forced to zero, and the terms consistent with zero value are then inferred. The last section of this note covers the valuation of a swap.

Let us begin with notation. In order to handle currency swaps, we shall price interest rate swaps in two countries, which we refer to as domestic and foreign. Let $r_D(0, d_1), r_D(0, d_2), ... r_D(0, d_n)$ be the domestic term structure of interest rates, representing rates for loans of $d_1, d_2, ..., d_n$ days. Let $r_F(0, d_1), r_F(0, d_2), ..., r_F(0, d_n)$ be the foreign term structure of interest rates. These are LIBOR-style rates, meaning that when interest is paid it is added on to the principal and adjusted by multiplying by $d_i/360$ where $d_i$ is the day count over the period or assuming 30 days in a month, whichever method the counterparties to the swap choose. For example, if one unit of currency is put in a LIBOR time deposit for $d_1$ days at the rate $r_D(0, d_1)$, the deposit grows to a value of $1[1 + r_D(0, d_1)(d_1/360)]$ or using $r_F$ as appropriate.

We can simplify the notation for swap pricing considerably by working with present value factors. By definition, a present value factor is the price of a zero coupon bond with a face value of one unit of currency. For a domestic bond maturing in $d_i$ days that pays interest in the LIBOR manner, its price is found as $B_D(0, d_i) = 1/[1 + r_D(0, d_i)(d_i/360)]$. A corresponding specification holds for a foreign bond. By definition, the forward rates are given as $r_D(0, d_a, d_b) = \{[1 + r_D(0, d_b)(d_b/360)]/[1 + r_D(0, d_a)(d_a/360)] - 1\}(360/(d_b - d_a)) - 1$ for the domestic term structure and a corresponding formula for the foreign term structure. The $0, d_a, d_b$ arguments on the forward rate indicate that the information is as of time 0 and represents a loan beginning at day $d_a$ and ending at day $d_b$, thus having a maturity of $d_b - d_a$. If $d_a = 0$ then the rate $r(0, 0, d_b)$ is expressed as $r(0, d_b)$, which is a spot rate.

The Appendix contains numerical examples that will be used in illustrating the results demonstrated herein.

In addition, TN05-01, 12-01, and 13-01 dovetail with this note. TN05-01 covers floating-rate securities, TN12-01 covers swaps with amortizing notional principals, and TN13-01 covers swaps in which the floating rate adjusts multiple times during the settlement period. It would, however, be best read this note before 12-01 and 13-01.
though you might benefit from reading 05-01 before or more or less concurrently with this one.

**Pricing Plain Vanilla Interest Rate Swaps**

Consider a swap with \( n \) specified payments. One party pays a floating rate and the other pays a fixed rate, which we designate as \( f \). Pricing the swap means to find the fixed rate such that the present value of the fixed payments equals the present value of the floating payments. Clearly the challenge is to find the present value of the floating payments. At first glance this may appear to be a formidable task but in comparison to the difficulty of pricing a stock, which has unknown floating payments, a swap is a remarkably simple instrument the pricing of which is aided by the existence of a term structure of interest rates, which contains an enormous amount of information. The first floating payment is determined by the spot rate \( r(0,d_1) \). The next floating payment will be \( r(d_1,d_2) \) and so on until the last, \( r(d_{n-1},d_n) \). Of course, all of the floating payments after the first one are not known at the time we are pricing the swap, but for the remaining floating payments we can use the one-period forward rates. Thus, seen from time 0, the second payment is set to be \( r(0,d_1,d_2) \), the third at \( r(0,d_2,d_3) \), etc. The actual payments will, of course, turn out to be whatever they are, based on changes in the term structure. *This procedure in no way invokes the expectations theory. These forward rates are not expectations of future spot rates.* Rather, these rates are arbitrage-free values. A party making floating payments can hedge the risk by substituting forward contracts that lock in the commitment to pay the forward rates currently embedded in the term structure.\(^4\) As such, we can use the forward rates as substitutes for the future spot rates.

At time 0 the value of the floating payments is

\[
V_{FL}(0) = \sum_{i=1}^{n} r(0,d_{i-1},d_i) \left( \frac{d_i - d_{i-1}}{360} \right) B(0,d_i).
\]

Thus, the value of the first payment is \( r(0,d_1)((d_1 - d_0)/360)B(0,d_1) \), the second is worth \( r(0,d_1,d_2)((d_2 - d_1)/360)B(0,d_2) \), etc.

The value of the fixed payments at time 0 is

\[
V_{FX}(0) = \sum_{i=1}^{n} f \left( \frac{d_i - d_{i-1}}{360} \right) B(0,d_i) = f \sum_{i=1}^{n} \left( \frac{d_i - d_{i-1}}{360} \right) B(0,d_i).
\]

\(^4\)These forward rates are expectations in the sense of equivalent martingale theory, a topic taken up in a different teaching note.
Since $f$ is the only unknown, the solution is easily found by setting $V_{FL}(0) = V_{FX}(0)$ and solving for $f$. Let us, however, obtain the value in a much simpler manner.

The floating payments in a swap can be viewed like a floating rate bond, but a floating rate bond contains a principal repayment at time $n$, which is not contained in the floating payments on a swap. Likewise, the fixed payments in a swap can be viewed like a fixed rate bond, which also involves a principal repayment at time $n$, though the fixed payments in a swap do not contain a principal repayment. But we can pretend that both the fixed and floating payments in the swap contain a principal repayment. These principal payments would offset so the combination of issuing (buying) a fixed-rate bond and buying (issuing) a floating-rate bond replicates a pay-fixed (-floating), receive-floating (-fixed) swap.\(^5\) Adding the principals, however, facilitates pricing. The value of the floating-rate bond is, therefore,

$$V_{FLRB}(0) = \sum_{i=1}^{n} r(0,d_{i-1},d_{i}) \left( \frac{d_{i} - d_{i-1}}{360} \right) B(0,d_{i}) + B(0,d_{n}).$$

The value of the fixed-rate bond is

$$V_{FXRB}(0) = f \sum_{i=1}^{n} \left( \frac{d_{i} - d_{i-1}}{360} \right) B(0,d_{i}) + B(0,d_{n}).$$

The value of a floating rate bond when initially offered equals par of 1.0, a result of the fact that the first coupon is set to equal the one-period market discount rate and of the assumption that the floating payment at each reset date will equal the then-current one-period discount rate.\(^6\) Since the present value of the fixed payments must equal the present value of the floating payments, the swap fixed rate can be found as the solution, $f$, to

$$1.0 = f \sum_{i=1}^{n} \left( \frac{d_{i} - d_{i-1}}{360} \right) B(0,d_{i}) + B(0,d_{n}),$$

which is an easily solvable problem. The fixed rate is

\(^5\)That is, if party A enters into a swap with party B to pay fixed and receive floating, the two parties can be viewed as having exchanged equivalent notional principals at the start and at the end of the swap. Obviously, this device ignores credit risk because a party could default, but that is a topic that we do not address here.

\(^6\)It is simple to prove this point. Simply substitute the definition of the forward rate, in terms of the corresponding spot rates, for the appropriate floating payment. Cancellation of appropriate terms leaves a value of 1.0.
\[ f = \frac{1.0 - B(0, d_n)}{\sum_{i=1}^{n} \left( \frac{d_i - d_{i-1}}{360} \right) B(0, d_i)} , \]

and is equivalent to finding the fixed rate on a par value fixed rate bond.

In most real situations, the distance between floating payments is the same and therefore \( d_i - d_{i-1} \) will always be the same regardless of the value of \( i \). That being the case, we designate \( d = d_i - d_{i-1} \). So our formula becomes

\[ f = \frac{1.0 - B(0, d_n)}{\sum_{i=1}^{n} B(0, d_i) \left( \frac{360}{d} \right)} . \]

See the Appendix for examples of calculations of the swap fixed rate.

**Pricing Currency Swaps**

Now let us price the four possible currency swaps. We can have fixed payments in the domestic currency vs. fixed payments in the foreign currency, floating payments in the domestic currency vs. floating payments in the foreign currency, and fixed payments in either vs. floating payments in the other. Let \( S_D(0) \) be the spot exchange rate, stated as units of the domestic currency per unit of foreign currency. We assume a one unit notional principal for the dollar payments. Nearly all currency swaps involve the exchange of notional principals at the beginning and at the end of the swap, but we shall also evaluate currency swaps without the exchange of notional principals.

First consider the fixed-fixed swap, assuming payment of the domestic currency and receipt of the foreign currency. We know that by using the domestic fixed rate, \( f_D \), the present value of the fixed payments is one unit of the domestic currency, \( 1_D \). Obviously, the present value of the foreign fixed payments is \( 1_F \). To equate these present value streams, we need only adjust the notional principal of the foreign payments to \( 1_F / S_D(0) \), a value we denote as \( N_{SW} \). Thus, the initial exchange of cash in the two currencies has equivalent value at the start.

For the floating-floating swap, the present value of the domestic payments is, again, one unit of the domestic currency, \( 1_D \). The present value of the foreign payments

---

\(^7\)We will encounter situations involving valuation of the swap during its life when this simplifying condition will not hold.
is, again, 1\(_F\). Again, by adjusting the foreign notional principal to \(N_F = 1_F/S_D(0)\), the present value of the foreign payments in domestic currency is 1\(_D\).

Based on these results, it is easy to see that with either party paying fixed and the other paying floating, the notional principals will be 1\(_D\) and \(N_F = 1_F/S_D(0)\). See the Appendix for sample calculations.

The logic behind the initial and final exchange of principals in a currency swap is that the transaction is a simplified representation of the process of issuing a (fixed- or floating-rate) bond in one currency and using the proceeds to purchase a (floating- or fixed-rate) bond in another currency. The initial exchange of principals has equivalent value, but the final exchange does not have equivalent value, assuming an exchange rate change. Some currency swaps omit the initial and final exchange of principals. As we show here, this affects the pricing. Consider the receive-fixed domestic vs. pay-fixed foreign swap. The present value of the domestic payments is

\[V_D(0) = f_D \sum_{i=1}^{n} \left( \frac{d_i - d_{i-1}}{360} \right) B_D(0, d_i),\]

and the present value of the foreign payments is

\[V_F(0) = f_F \sum_{i=1}^{n} \left( \frac{d_i - d_{i-1}}{360} \right) B_F(0, d_i).\]

To equate these two, we need only multiply \(V_F\) by the exchange rate \(S_D(0)\) times the foreign notional principal, \(N_F\). In other words,

\[N_F S_D(0) V_F(0) = V_D(0),\]

and the solution is found by solving for the one unknown, \(N_F\).

A similar approach is found in the floating domestic-floating foreign swap. The answer, \(N_F\), is the same as the fixed-fixed case. Similarly, the answer is the same for the floating domestic-fixed foreign and fixed domestic-floating foreign swaps. Examples are provided in the Appendix.

**Pricing Interest Rate Swaps as Forward Contracts**

It is well known that interest rate swaps can be modeled as combinations of forward rate agreements or FRAs. An FRA is a forward contract on an interest rate and is essentially a swap with one payment. The fixed rate on an FRA is set at the appropriate forward rate. In practice an FRA settles at the point of expiration. For

...
example, a 3x6 FRA (pronounced “3 by 6”) on LIBOR is a forward contract calling for one party to pay the 90-day LIBOR that prevails three months from now with the other party paying a fixed rate and payments occurring in three months. In a swap, and almost all other interest rate derivatives, the payment is deferred. For example, a one-year swap on 90-day LIBOR with settlement each quarter will involve the payment of 90-day LIBOR vs. a fixed rate where the LIBOR is set at the beginning of the 90-day period and both payments are made at the end of the period.

To replicate a swap by FRAs, one would combine a series of FRAs with expirations corresponding to the swap settlement dates but where the FRA payments are made at the end of the settlement period. In a swap, however, all of the fixed payments are the same, while in an FRA, each fixed payment would differ, assuming a non-flat term structure. Thus, a swap is actually a series of off-market FRAs. Also, note that since the first floating payment in a swap is known when the swap is initiated, the first payment is simply an exchange of known but different cash flows and not technically an FRA.

Consider a pay-fixed, receive-floating swap of n payments. Decomposing the swap into a series of off-market FRAs, we can express the value of the swap as

$$V_{SWAP}(0) = \sum_{i=1}^{n} V_{FRA}(0,d_{i-1}, f)$$,

where $V_{FRA}(0,d_{i-1}, f)$ is the value at time 0 of an FRA expiring on day $d_{i-1}$, with payment on day $d_i$ where the fixed payment is $f$, the swap fixed rate, and the floating payment is $r(0,d_{i-1},d_i)$, both adjusted by the day count factor, $(d_i - d_{i-1})/360$. As of the initiation date of the swap, the floating rates are conditionally set to the forward rates, $r(0,d_{i-1},d_i)$; they will, of course, change later as interest rates change. Thus, $V_{FRA}(0,d_{i-1}, f) = [r(0,d_{i-1},d_i) - f][(d_i - d_{i-1})/360]B(0,d_i)$. As noted above, since the first floating payment is known, it is technically not an FRA, but the notation $[r(0,d_{0},d_{1}) - f] ((d_{1} - d_{0})/360)B(0,1)$ still correctly specifies the value of that payment. See the Appendix for examples.

**Pricing Currency Swaps as Forward Contracts**

In a similar manner, a currency swap can be priced as a combination of currency forward contracts, with, however, the first contract being simply an exchange of known

---

8Keep in mind that days $d_i$ and $d_{i-1}$ are not one day apart. For example, if $i = 2$, then $d_2$ and $d_1$ are not day 2 and day 1. These expressions refer to the days of the second and first payments.
cash flows at the current exchange rate. To value the swap as a combination of currency forwards, we require the forward exchange rates. Recalling that $S_D(0)$ is the spot exchange rate, let $S_D(0,d_i)$ be the forward exchange rate observed at time 0 for day $d_i$. When $i = 0$, then $S_D(0,0) = S_D(0)$.

Now consider the fixed domestic-fixed foreign swap. In units of domestic currency, its value can be expressed as

$$V_{FXFXCS}(0) = \sum_{i=1}^{n} V_{CF}(0,d_i,f_i,F_i,f_i,D_i),$$

where $V_{CF}(0,d_i,f_i,F_i,f_i,D_i)$ is the value of a currency forward expiring on day $d_i$ involving the payment of a fixed rate in domestic currency and the receipt of a fixed rate in foreign currency. It is given as $V_{CF}(0,d_i,f_i,F_i,f_i,D_i) = [f_i,N_i f S_D(0) - f_i D_i]B_D(0,d_i)$. For $d_i = d_n$, $V_{CF}(0,d_n,f_n,F_n,f_n,D_n) = [(1+f_n F_i)N_T S_D(0) - (1+f_n D_i)]B_D(0,d_n)$, reflecting the repayment of both principals. Also, at the start there is an exchange of $N_F S_D(0)$ against $1_D$, but of course there is no net value associated with this initial exchange of principals.

For the floating domestic-floating foreign swap, the value can be expressed as

$$V_{FXFXCS}(0) = \sum_{i=1}^{n} V_{CF}(0,d_i,r_i,F_i,r_i,D_i),$$

which indicates that the rate $r_D(0,d_{i-1},d_i)$ will be paid and the rate $r_F(0,d_{i-1},d_i)$ will be received. Here the component currency forward involves a floating payment on both sides and its value can be expressed as $V_{CF}(0,d_i,r_i(0,d_{i-1},d_i),r_F(0,d_{i-1},d_i)) = [r_F(0,d_{i-1},d_i)N_F S_D(0) - r_D(0,d_{i-1},d_i)]B_D(0,d_i)$. Naturally there is a valueless initial exchange of principals, and the forward contract at time $n$ would need to include the exchange of $N_F$ units of the foreign currency for one unit of the domestic currency.

Likewise, the remaining currency swaps (domestic fixed-foreign floating and domestic floating-foreign fixed) can be valued in a similar manner, using either the appropriate floating rate $r(0,d_{i-1},d_i)$ or the fixed rate for the domestic and foreign payments. For currency swaps with no notional principal exchange, slight adjustments are required to remove the notional principals at the beginning and at the end. See the appendix for numerical examples.

Swaps can also be priced as combinations of call and put options. A LIBOR pay-fixed, receive-floating swap can be priced as a combination of long calls and short puts on LIBOR with equivalent exercise rates and where the exercise rate equals the fixed rate.
on the corresponding swap. Likewise, a currency swap can be viewed as a combination of currency options. Naturally pricing these options would require an arbitrage-free term structure model.

**Valuation of Swaps**

Now that we have identified the terms that permit a swap to have zero value at the start, we can proceed to understand how the swap value changes during the life of the swap. This process is called *valuation*. The value of a swap is simply the amount it is worth. To value a swap is nothing more than to determine the present value of the promised inflows net of the present value of the promised outflows. This value is sometimes referred to as *replacement value*, mostly in the context of the cost to the holder of a swap that has positive value in the event that the counterparty defaults.

Valuation of a swap is quite easy. After the swap is established, the terms, except for any promised floating payments, are fixed. To value a plain vanilla swap, one simply discounts the fixed payments using the new interest rates. One then discounts the floating payments using one of two procedures that correspond to those we identified in the pricing section. We can either add the notional principal to the floating payments and the fixed payments or simply solve for the new forward rates and use those as the unknown future floating payments.

Suppose we are at day $t$, which is after day 0 but before day $d_1$. We have a new term structure of interest rates, $r(d_i,d_{i+1})$, $r(d_0,d_1)$, etc., and a new set of discount bond prices, $B(d_i,d_{i+1})$, $B(d_0,d_1)$, etc., for either fixed or foreign rates. Note, however, that a rate like $r(d_0,d_1)$ is a rate spanning a period of less than $d_1 – d_0$ days. The present value at day $t$ of the fixed payments that were previously set at the rate $f$, is obviously

$$V_{fix}(t) = f \sum_{i=1}^{n} \left( \frac{d_i - d_{i-1}}{360} \right) B(d_i,d_{i+1}).$$

Finding the present value of the floating payments is only slightly more difficult. The upcoming floating payment was fixed at the last reset date, day 0 here, at $r(d_0,d_1)$. All remaining floating payments are yet to be determined. In the section on pricing, we demonstrated that a swap can be shown to be a series of FRAs. Consequently, we can use the new forward rates (at time $t$), obtained using the new spot rates, as our floating payments. For $i = 1$, the known upcoming payment, we have $r(d_0,d_1)(d_1 –
For \( i = 2 \), the second floating payment, which is currently unknown, we use \( r(d_0,d_1,d_2) \), which is the forward rate from day \( d_1 \) to day \( d_2 \) observed at day \( d_0 \), and we follow a similar pattern for \( i = 3, \ldots, n \). Thus,

\[
V_{FL}(t) = r(d_0,d_i)\left(\frac{d_i - d_0}{360}\right)B(d_i,d_i) + \sum_{i=2}^{n} r(d_i,d_{i-1},d_i)\left(\frac{d_i - d_{i-1}}{360}\right)B(d_i,d_i).
\]

Valuation of the swap is then easily found by subtracting the present value of one stream of payments from that of the other.

A simpler way to value the swap is to add the notional principal to both sides, as we did when pricing the swap. Treating the stream of payments as a fixed rate bond, the swap value is

\[
V_{FXRB}(t) = f \sum_{i=1}^{n} \left(\frac{d_i - d_{i-1}}{360}\right)B(d_i,d_i) + B(d_i,d_n).
\]

This trick will greatly facilitate finding the present value of the floating payments. As previously noted, the present value of the floating payments plus a final principal will always be par when the next payment is reset to the current market interest rate for that upcoming period. Consequently, at the next payment date we would receive the next known floating payment plus a claim on the remaining unknown floating payments and the final principal, which at that time must have a value of one unit of currency, the standard notional principal. Thus, treating the stream of payments as a floating rate bond, the value of the floating payments on the swap is

\[
V_{FLRB}(t) = (1 + r(d_0,d_i))\left(\frac{d_i - d_0}{360}\right)B(d_i,d_i).
\]

There is yet another way to value a swap, which is to determine the fixed rate on a new swap that would offset the old swap and treat the new swap as a replacement for the old swap. We must, however, proceed in a very careful manner. Recall that we are at day \( d_i \) between day \( d_0 \) and day \( d_1 \), and the term structure consists of the rates \( r(d_0,d_1), r(d_1,d_2), \ldots, r(d_n,d_n) \). The upcoming floating payment has already been established as \( r(d_0,d_1) \). A new swap with the same payment dates as the old swap will have a different floating rate, \( r(d_0,d_1) \) and a new fixed rate, which we will denote as \( f^* \). If the new swap is entered into while maintaining the old swap, all of the floating rates except the upcoming one will offset. Of course, the upcoming floating rates on both the old and new swaps are
not actually floating, inasmuch as they have already been fixed. ⁹ We will see, therefore, that this new swap will eliminate all of the unknown floating rates, and therefore, all of the uncertainty, leaving only fixed payments that can be easily valued.

For example, suppose the old swap calls for fixed payments of \( f \) at days \( d_1, d_2, \ldots, d_n \) and floating receipts at days \( d_1, d_2, \ldots, d_n \). The upcoming floating receipt has been set at \( r(d_0, d_1) \). So the floating receipts will be \( r(d_0, d_1) \), which is known, and \( r(d_1, d_2), r(d_2, d_3), \ldots, r(d_{n-1}, d_n) \), which are unknown. A new swap designed to pay floating and receive fixed will have a series of floating payments of \( r(d_i, d_{i+1}) \) and fixed receipts of \( f^* \) at days \( d_1, d_2, \ldots, d_n \). Ignoring the day count adjustment, this new swap combined with the old swap will result in payments of

- At day \( d_1 \): pay \( f \), receive \( f^* \), receive \( r(d_0, d_1) \), pay \( r(d_1, d_1) \)
- At day \( d_2 \): pay \( f \), receive \( f^* \), receive \( r(d_1, d_2) \), pay \( r(d_1, d_2) \)
- At day \( d_3 \): pay \( f \), receive \( f^* \), receive \( r(d_2, d_3) \), pay \( r(d_2, d_3) \)

...  
- At day \( d_n \): pay \( f \), receive \( f^* \), receive \( r(d_{n-1}, d_n) \), pay \( r(d_{n-1}, d_n) \)

Note how all of the unknown floating payments offset, leaving only known payments of the following:

- At day \( d_1 \): \( f^* - f, r(d_0, d_1) - r(d_1, d_1) \)
- At day \( d_2 \): \( f^* - f \)
- At day \( d_3 \): \( f^* - f \)

...  
- At day \( d_n \): \( f^* - f \)

Thus, the value of the old swap can be easily obtained as the present value of the above stream of known payments. Formally,

\[
V(t) = (r(d_0, d_1) - r(d_i, d_j)) \left( \frac{d_i - d_0}{360} \right) B(d_i, d_j) + (f^* - f) \sum_{i=2}^{n} \left( \frac{d_i - d_{i-1}}{360} \right) B(t, i).
\]

To value a currency swap, we follow the same procedures, determining the value in the respective currencies of each set of payments. We then convert the foreign currency to the domestic currency using the new current spot exchange rate.

---

⁹Remember that the old swap set the upcoming floating payment at time 0 as \( r(d_0, d_1) \). A new swap set up at time \( t \) will set the upcoming floating payment at \( r(d_i, d_{i+1}) \). As of time \( t \), these rates are known.
Alternatively, we could value the currency swap as a series of currency forward contracts and determine their overall net value.

Calculations in the appendix demonstrate how swaps are valued.

Appendix

This appendix illustrates the results presented in this teaching note with numerical examples involving a three-period swap. We will use the U. S. dollar as the domestic currency and the Swiss franc as the foreign currency and use the symbols US and SF instead of D and F. Let the following information about the U. S. and Swiss structures be given. (The subscripts are omitted here but used later where necessary).

<table>
<thead>
<tr>
<th>U. S. Term Structure</th>
<th>Swiss Term Structure</th>
</tr>
</thead>
<tbody>
<tr>
<td>( r(0,1) = 0.08, r(0,2) = 0.09, r(0,3) = 0.10 )</td>
<td>( r(0,1) = 0.088, r(0,2) = 0.093, r(0,3) = 0.105 )</td>
</tr>
<tr>
<td>( B(0,1) = 1/[1 + 0.08(360/360)] = 0.9259 )</td>
<td>( B(0,1) = 1/[1 + 0.088(360/360)] = 0.9191 )</td>
</tr>
<tr>
<td>( B(0,2) = 1/[1 + 0.09(720/360)] = 0.8475 )</td>
<td>( B(0,2) = 1/[1 + 0.093(720/360)] = 0.8432 )</td>
</tr>
<tr>
<td>( B(0,3) = 1/[1 + 0.10(1080/360)] = 0.7692 )</td>
<td>( B(0,3) = 1/[1 + 0.105(1080/360)] = 0.7605 )</td>
</tr>
</tbody>
</table>

The implied forward rates for one-period transactions are

(U. S.)
\[
\begin{align*}
\hat{r}(0,1,2) &= \left( \frac{[1 + 0.09(720/360)]}{[1 + 0.08(360/360)]} \right) - 1 \times 360/360 = 0.0926 \\
\hat{r}(0,2,3) &= \left( \frac{[1 + 0.10(1080/360)]}{[1 + 0.09(720/360)]} \right) - 1 \times 360/360 = 0.1017
\end{align*}
\]

(Swiss)
\[
\begin{align*}
\hat{r}(0,1,2) &= \left( \frac{[1 + 0.093(720/360)]}{[1 + 0.088(360/360)]} \right) - 1 \times 360/360 = 0.0901 \\
\hat{r}(0,2,3) &= \left( \frac{[1 + 0.105(1080/360)]}{[1 + 0.093(720/360)]} \right) - 1 \times 360/360 = 0.1088
\end{align*}
\]

Let the spot exchange rate be \( S_{US}(0) = $0.70 \). The forward exchange rates can be easily derived as
\[
\begin{align*}
S_{US}(0,1) &= 0.70 \times \frac{1 + 0.08(360/360)}{1 + 0.088(360/360)} = $0.6949 \\
S_{US}(0,2) &= 0.70 \times \frac{1 + 0.09(720/360)}{1 + 0.093(720/360)} = $0.6965 \\
S_{US}(0,3) &= 0.70 \times \frac{1 + 0.10(1080/360)}{1 + 0.105(1080/360)} = $0.6920,
\end{align*}
\]

which, of course, one obtains from applying the interest rate parity rule.

Pricing Plain Vanilla Swaps

---

10These values were calculated in Excel so due to computational precision, they may differ slightly than what you would obtain if you insert them into a calculator.
Now let us price the plain vanilla swap in both countries. For the domestic swap, the present value of the floating payments is

\[ V_{FL}(0) = 0.08(0.9259) + 0.0926(0.8475) + 0.1017(0.7692) = 0.2308, \]

which is set equal to the present value of the fixed payments:

\[ V_{FX}(0) = f_{US}(0.9259 + 0.8475 + 0.7692) = 0.2308. \]

Solving for the fixed payment gives \( f_D = 0.0908 \). As an alternative, we could simply note that the present value of the floating payments plus a final principal is \( 0.2308 + 1(0.7692) = 1.0000 \), as it should be. Then equating the present value of the fixed payments plus a final principal, we have \( f_D(0.9259 + 0.8475 + 0.7692) + $1(0.7692) = 1.0000 \). Solving for \( f_D \) again gives \( 0.0908 \). Of course, this alternative method avoids having to compute the forward rates.

For the foreign swap, the present value of the floating payments is

\[ V_{FL}(0) = 0.088(0.9191) + 0.0901(0.8432) + 0.1088(0.7605) = 0.2396, \]

and equating to the present value of the fixed payments, we have

\[ V_{FX}(0) = f_F(0.9191 + 0.8432 + 0.7605) = 0.2396. \]

Solving gives \( f_F = 0.0950 \). Likewise, adding a principal repayment gives a present value of the floating payments as \( 0.2395 + 0.7605 = 1.0000 \) and a present value of the fixed payments set at \( f_F(0.9191 + 0.8432 + 0.7605) + 0.7605 = 1.0000 \). Again, \( f_F = .0950 \).

**Pricing Currency Swaps**

Now let us price the currency swap involving payment of dollars and receipt of Swiss francs with an initial and final exchange of notional principals. First we price the fixed-fixed currency swap. With a domestic notional principal of one dollar, the foreign notional principal should be set at \( 1/$0.70 = SF1.4286 \). This is easily verified by finding the present values of both streams of payments and converting them to a common currency, here the dollar. We already know that the present value of the dollar fixed payments is $1. The present value of the Swiss franc payments on SF1.4286 notional principal is SF1.4286[.0950(0.9191) + .0950(0.8432) + .0950(0.7605) + 1(0.7605)] = SF1.4286. At a $0.70 exchange rate, this is equivalent to $1 at the start.

For the floating-floating currency swap, the results are the same. Given that we found the plain vanilla fixed rates in each country by equating the present value of the floating payments to the present value of the fixed payments, we know that a notional
principal of $1 and SF1.4286 will have the equivalent value, whether the payments are fixed or floating. Likewise, the currency swaps where one party will pay fixed and the other floating will have the same notional principals, $1 and SF1.4286.

For currency swaps with no initial or final exchange of principal, the pricing will be different. For the fixed-fixed swap, the present value of the dollar fixed payments is $0.0908(0.9259 + 0.8475 + 0.7692) = 0.2308. The present value of the Swiss franc fixed payments for an unspecified notional principal of $N_{SF}$ is $N_{SF}[0.0950(0.9191 + 0.8432 + 0.7605)]($0.70) = N_{DF}(0.1678). Equating this value to 0.2308 and solving for $N_{SF}$ gives a Swiss franc notional principal of SF1.3754. The same result would be obtained for the swap with floating-floating or one currency fixed and the other floating.

**Plain Vanilla Swaps as Combinations of Forward Rate Agreements (FRAs)**

Now let us value these swaps as combinations of forward contracts. For the plain vanilla swaps, we can value the US swap as the following combination of FRAs:

\[
V_{US}(0) = (0.08 - 0.0908)(0.9259) + (0.0926 - 0.0908)(0.8475) + (0.1017 - 0.0908)(0.7692) = -0.0100 + 0.0015 + 0.0084 \approx 0.0000,
\]

and for the Swiss plain vanilla swap, we have

\[
V_{SW}(0) = (0.088 - 0.0950)(0.9191) + (0.0901 - 0.0950)(0.8432) + (0.1088 - 0.0950)(0.7605) = -0.0064 + 0.0041 + 0.0105 \approx 0.0000.
\]

We see that in both cases, the component FRAs are not individually valued at zero, as they would be if removed separately and priced at their respective market rates. That is, each FRA would ordinarily be priced at the appropriate forward rate to make its value zero. Hence, we call these off-market FRAs. Also, the first payment is actually not an FRA but rather a spot transaction. Collectively these transactions have zero value for each party and in each respective country.

**Currency Swaps as Combinations of Currency Forwards**

Now let us look at the currency swaps as combinations of currency forwards. For the fixed-fixed currency swap, involving payment of dollars and receipt of Swiss francs, the value is found as

\[
V_{FXFXCS} = [0.0950(1.4286)(0.6949) - 0.0908]0.9259
\]
Given our equivalence between the fixed and floating payments in each country, it should be easy to see that the remaining swap values can be similarly decomposed into currency forwards. The component forward values will not match up individually with these three values, but the overall value of the combination of contracts will be zero.

**Valuation of Swaps**

Let us now assume that we are six months into the life of the swap. The new term structures are

<table>
<thead>
<tr>
<th>U. S. Term Structure</th>
<th>Switzerland Term Structure</th>
</tr>
</thead>
<tbody>
<tr>
<td>$r(0.5,1) = 0.082$, $r(0.5, 2) = 0.094$, $r(0.5, 3) = 0.105$</td>
<td>$r(0.5,1) = 0.09$, $r(0.5,2) = 0.096$, $r(0.5,3) = 0.108$</td>
</tr>
<tr>
<td>$B(0.5,1) = 1/[1 + 0.082(180/360)] = 0.9606$</td>
<td>$B(0.5,1) = 1/[1 + 0.09(180/360)] = 0.9569$</td>
</tr>
<tr>
<td>$B(0.5,2) = 1/[1 + 0.094(540/360)] = 0.8764$</td>
<td>$B(0.5,2) = 1/[1 + 0.096(540/360)] = 0.8741$</td>
</tr>
<tr>
<td>$B(0.5,3) = 1/[1 + 0.105(900/360)] = 0.7921$</td>
<td>$B(0.5,3) = 1/[1 + 0.108(900/360)] = 0.7874$</td>
</tr>
</tbody>
</table>

The new implied forward rates are

(U.S.)

$$r(0.5,1,2) = \left(\frac{\left[1 + 0.094(540/360)\right]}{\left[1 + 0.082(180/360)\right]} - 1\right)\frac{360}{360} = 0.0961$$

$$r(0.5,2,3) = \left(\frac{\left[1 + 0.105(900/360)\right]}{\left[1 + 0.094(540/360)\right]} - 1\right)\frac{360}{360} = 0.1065$$

(Switzerland)

$$r(0.5,1,2) = \left(\frac{\left[1 + 0.096(540/360)\right]}{\left[1 + 0.09(180/360)\right]} - 1\right)\frac{360}{360} = 0.0947$$

$$r(0.5,2,3) = \left(\frac{\left[1 + 0.108(900/360)\right]}{\left[1 + 0.096(540/360)\right]} - 1\right)\frac{360}{360} = 0.1101$$

Let the new spot exchange rate be $S_{US}(0) = $0.725. Recall that the original fixed rates were 0.0908 in the U. S. and 0.0950 in Switzerland. The original notional principals were $1 in the U. S. and SF1.4286.

First let us look at valuing domestic plain vanilla swaps. We take the position of a party paying fixed and receiving floating. Let us initially add the notional principals.
For a U. S. swap, the fixed payments can be treated like a fixed-rate bond, which will have a present value of

\[ V_{FXRB}(t) = 0.0908(0.9606) + 0.0908(0.8764) + 1.0908(0.7921) = 1.0307. \]

Remember that the present value of the floating payments, with notional principal included, is found as the present value of the next floating payment, which is known at time 0 to be the one-period spot rate of 0.08, plus the present value of par value of 1.0 at time 1. Remember that 1.0 at time 1 represents the value at time 1 of all remaining floating payments plus par value paid at time n. Thus, treating the floating payments like a floating-rate bond, we have

\[ V_{FLRB}(t) = 1.08(0.9606) = 1.0375. \]

Hence, a swap paying fixed and receiving floating is worth 1.0375 - 1.0307 = 0.0068. In Switzerland a SF1 notional principal plain vanilla swap would be found as

\[ V_{FXRB}(t) = 0.0950(0.9569) + 0.0950(0.8741) + 1.0950(0.7874) = 1.0360, \]
\[ V_{FLRB}(t) = 1.088(0.9569) = 1.0411; \]

thus, the swap value would be 1.0411 - 1.0360 = 0.0051 per SF1 notional principal.

Alternatively, we could value the swap as a combination of forward contracts, using the new forward rates to reflect the unknown future floating payments. Again, we repeat, this does not mean that the new forward rates are our expectations of the future floating rates. They simply reflect the substitutability of forward rates for future spot rates that is permissible when forward contracts can be perfectly hedged.

In the U. S. market, the present value of the fixed payments is

\[ V_{FX}(t) = 0.0908(0.9606 + 0.8764 + 0.7921) = 0.2386, \]

and the present value of the floating payments is,

\[ V_{FL}(t) = 0.08(0.9606) + 0.0961(0.8764) + 0.1065(0.7921) = 0.2454, \]

which uses the new forward rates for the second and third floating payments. The net difference is 0.0068.

In the Swiss market, the present value of the fixed payments is

\[ V_{FX}(t) = 0.0950(0.9569 + 0.8741 + 0.7874) = 0.2486, \]

and the present value of the floating payments is,

\[ V_{FL}(t) = 0.088(0.9569) + 0.0947(0.8741) + 0.1101(0.7874) = 0.2537. \]

The net difference is 0.0051, which is exactly what we previously obtained.
Valuation of the currency swap is then simple. Consider a currency swap without a notional principal involving the payment of U. S. dollars and receipt of Swiss francs. We previously determined that a $1 notional principal is equivalent to a SF1.4286 notional principal. The new spot exchange rate is $0.725. For the fixed-fixed swap, we pay dollars at the fixed rate of 0.0908 and receive Swiss francs at the fixed rate of 0.0950. The value of the swap is, therefore,

\[ V_{FXFX}(t) = 1.4286(0.2486)(0.725) - 0.2386 = 0.0189, \]

which uses the previously-determined present value of the Swiss francs fixed payments per SF1 notional principal of 0.2486 and present value of the dollar fixed payments of 0.2386. If we were paying dollars floating and receiving Swiss francs fixed, the swap value would be

\[ V_{FXFL}(t) = 1.4286(0.2486)(0.725) - 0.2454 = 0.0121, \]

which uses the previously-determined present value of the Swiss franc fixed payments per SF1 of 0.2486 and present value of the dollar floating payments of 0.2454. If we were paying dollars fixed and receiving Swiss francs floating, the swap value would be

\[ V_{FLFX}(t) = 1.4286(0.2537)(0.725) - 0.2386 = 0.0242, \]

which uses the previously-determined present value of the Swiss franc fixed payments per SF1 of 0.2537 and present value of the dollar fixed payments of 0.2386. If we were paying dollars floating and receiving Swiss francs floating, the swap value would be

\[ V_{FLFL}(t) = 1.4286(0.2537)(0.725) - 0.2454 = 0.0174, \]

which uses the previously-determined present value of the Swiss franc floating payments of 0.2537 per SF1 and present value of the dollar floating payments of 0.2454. These swap values can also be determined by valuing the component payments as currency forward contracts. The notional principals are added if notional principal would be paid on that swap.

As described in the main body of this note, a swap can also be valued using the concept of replacement value. That is, an old swap can be offset by a new swap. We construct the new swap, using its floating payments to offset the floating payments on the old swap, thereby leaving only a set of fixed payments, which are naturally easy to value. We now show how this works with this example. For the U. S. dollar swap, recall that we start with a three-year swap in which the fixed rate is 9.08%. We then move 0.5 years
forward and face a new term structure. We have not yet determined the fixed rate on a brand new swap so let us do so now. Recall that the discount factors are 0.9606 for 0.5 years, 0.8741 for 1.5 years, and 0.7921 for 2.5 years. In applying this new term structure to solve for the fixed rate on a new swap, there is one small wrinkle. Most swaps start off with payments equally spaced, but in this case we need the payments to occur 0.5, 1.5, and 2.5 years from now. So the first payment is one-half year from now, but the second and third payments are separated by a full year. The formula we learned for the fixed rate on a swap must, therefore, be adjusted. Since the second and third payments are annual, we adjust them technically by 360/360. The upcoming payment will have to be adjusted by 180/360. Therefore, the present value of the fixed payments, plus the hypothetical notional principal, is still

\[ V_{\text{FRBB}}(t) = f \sum_{i=1}^{n} \left( \frac{d_i - d_{i-1}}{360} \right) B(d_i, d_i) + B(d_i, d_n). \]

But the values \( d_i - d_{i-1} \) are not all the same. Nonetheless, we will find the fixed rate that forces this value to equal 1. Thus, our overall is the same as before, but we cannot apply it without some care. We have to use a different value of \( d_i - d_{i-1} \) for \( i = 1 \) than for \( i > 1 \).

\[ f = \frac{1.0 - B(0, d_n)}{\sum_{i=1}^{n} \left( \frac{d_i - d_{i-1}}{360} \right) B(0, d_i)}. \]

Thus, in this example,

\[ f = \frac{1.0 - 0.7921}{0.9606(180 / 360) + 0.8764 + 0.7921} = 0.0968. \]

Notice that the 180/360 is applied to the first payment while 360/360 is implicitly applied to the remaining payments. Now recall that if we used this new swap to offset the old one, it would leave us with receiving the upcoming floating payment on the old swap (0.08), paying the upcoming fixed payment on the old swap (0.0908), paying the upcoming floating payment on the new swap (0.082), receiving the upcoming fixed payment on the new swap (0.0968), and then receiving the difference between the upcoming fixed payment on the new swap (0.0968 received) and the upcoming fixed payment on the old swap (0.0908 paid) exactly 1.5 and 2.5 years from now. Thus, the value of the swap is
\[ V(0.5) = (0.0968 - 0.082) \left( \frac{180}{360} \right) 0.9606 + (0.08 - 0.0908)0.9606 \]
\[ + (0.0968 - 0.0908)(0.8764 + 0.7921) \]
\[ = 0.0068. \]

This is the value we previously obtained. For the Swiss franc swap, the new fixed rate is
\[ f = \frac{1.0 - 0.7874}{0.9569(180/360) + 0.8741 + 0.7874} = 0.0993. \]

The value of the swap is, therefore,
\[ V(0.5) = (0.0993 - 0.09) \left( \frac{180}{360} \right) 0.9569 + (0.0880 - 0.0950)0.9569 \]
\[ + (0.0993 - 0.0950)(0.8741 + 0.7874) \]
\[ = 0.0049. \]

This differs from the value we previously obtained (0.0051) by only a round-off error.

**References**

A variety of sources cover swap pricing.


Corb, H.  *Interest Rate Swaps and Other Derivatives* (New York: Columbia University Press, 2012), Ch. 3.


An interesting article showing the parity relations among interest rate and currency swaps and options is

An alternative view of pricing swaps is provided in


Jarrow and Turnbull (JT) argue that you cannot add the principals to both sides of a plain vanilla swap because the two streams of payments must be discounted at different rates, the floating being discounted at LIBOR and the fixed being discounted at the Treasury bill (t-bill) rate. The majority view is consistent with the approach taken in this teaching note and not with their position. JT’s view seems to incorporate market and credit risk, which is the primary factor that distinguishes t-bill from LIBOR rates. JT argue that the risks of the streams are different but if that is the case, then JT themselves discount improperly. On the floating side, they find the present value of the floating payments by subtracting the present value of the principal from the value of a floating rate security by discounting the principal at LIBOR. Yet, if they treat fixed payments as appropriately discounted at the t-bill rate, then the present value of the fixed principal repayment should be discounted at the t-bill rate. Moreover, JT do not incorporate credit risk into valuation of other derivatives. Hence, I disagree with position they take in this book.