Empirical Topics in Finance
Properties of Prices

Jon Danielsson, London School of Economics
Dek Terrell, Louisiana State University
This section follows Campbell Loo and Mackinlay 97. (CLM) ch 1. and 2. And Fama 1991. The treatment is however different, especially many of the technical details will be skipped. As always, the relevant material is the notes. CLM is only background reading. Also note that many definitions can be found in the appendix.

1 Introduction

The purpose of this chapter is the exploration of what prices of financial assets are, what are their properties, economically, statistically, and financially. As such, the chapter is the introduction to the other material in the course, and no detailed examination of the critical issues will be presented. The broad outline is that first we define prices and the efficient market hypothesis (EM), then the basic calculations, and conclude with some discussion of technical trading.

2 Efficient Markets

Generally, a capital market are said to be efficient if prices are determined by supply and demand in a competitive market filled with rational traders. These rational traders use any and all relevant information, which causes supply, demand, and the price of an asset to reflect information. As a consequence, there should be no opportunities to earn abnormal profits because prices reflect all relevant information.

Market efficiency does not imply equal expected returns to assets, because expected returns may differ across assets to compensate for differences in risk across assets. However, market efficiency does imply that no asset should earn an expected return exceeding a fair payment for the risk of that asset.

This classic quote from Malkiel provides a more succinct definition of market efficiency:

A capital market is said to be efficient if it fully and correctly reflects all relevant information in determining security prices. Formally, the market it said to be efficient with respect to some information set... if security prices would be unaffected by revealing that information to all participants. Moreover, efficiency with respect to an information set implies that it is impossible to make economic profits by trading on the basis of [that information set]

So, from our point of view:

1. prices reflect information
2. efficiency is testable
3. you can’t make money

Jon Danielsson ©2000
Alternatively one could use Roberts (1967) classifications of market efficiency, expanded in Fama 91.

1. **Return Predictability** All information in historical prices is reflected in current prices. Tests of the form of market efficiency are possible by examining the returns from market strategies based on time series prediction, technical trading, or other forecasts created based only on historical prices.

2. **Semi–strong** All publicly available information is fully reflected in current prices, announcements. Event studies can be used to examine semi-strong efficiency.

3. **Strong form** All information is reflected in prices. This is of course hard to access, given that insider trading is illegal.

Notice that these definitions of market efficiency differ in one key area, the information set available to traders and reflected in prices. For the remainder of this book, we will use $\Omega$ to denote an information set.

Does market efficiency imply that prices should be smooth? No, because prices change to reflect new information flowing into the market at any point in time. The release of an earnings report, announcement of a new product, or a host of other factors brings new information to the market. Prices will continuously adjust to reflect these information flows which will typically arrive in a very irregular manner. Thus, ex post returns may differ substantially across assets with similar risk characteristics. Market efficiency simply tells us that abnormal returns could not be predicted in advance. Of course, prices change to reflect new information, so actual returns may be much different from what even rational traders predicted.

### 2.1 Linear Relation and Expectations

The simplest economic model for asset prices models the relationship between asset prices and dividends. The logic is quite easy.

If the simple return on an asset is

$$R_{t+1} = \frac{P_{t+1} + D_{t+1}}{P_t} - 1$$

where $R$ is the return, $P$ the price, and $D$ the dividend, then

$$P_t = E_t \left[ \frac{P_{t+1} + D_{t+1}}{1 + R} \right]$$

Equation 2 simply states that the price at time $t$ should be equal to the the expected price next period plus expected dividends discounted into time $t$ dollars.

This result will also hold in period $t+1$ so,

$$P_{t+1} = E_{t+1} \left[ \frac{P_{t+2} + D_{t+2}}{1 + R} \right]$$

A more general result requires a bit of algebra using the law of iterated expectations.

Jon Danielsson ©2000
Definition 1 (Law of Iterated Expectations) If an information set \( \Omega_1 \) is larger \( \Omega_2 \) in the sense \( \Omega_2 \subset \Omega_1 \) then it is sufficient to write expectation in terms of \( \Omega_1 \)

\[
E [X|\Omega_1] = E [E [X|\Omega_2]|\Omega_1]
\]

Using the law of iterated expectations and substituting equation (3) (and similar pricing equations for all future prices) into equation (2) leads to the conclusion that prices are given by the present value of all future expected dividends.

\[
P_t = P_{Dt} = E_t \left[ \sum_{i=1}^{\infty} \left( \frac{1}{1+R} \right)^i D_{t+i} \right]
\]

(4)

Our basic economic model tells us that the current value of an asset is the present discounted value of all future receipts from that asset. For a stock, this implies that expected future prices reflect expected dividends, and if that expectation if good we get the empirically observed fact that there is some predictability in returns over long horizons. This also implies that the value of the firm is related to some fundamentals, specifically future earnings.

Technical aside

The constant expected stock return model is known as the martingale model of stock prices, but it does not imply that prices are a martingale. But the prices have a unit root if \( D \) is linear with a unit root. Therefore we have two unit root processes \( P \) and \( D \). Even if both are non-stationary, they are cointegrated.

2.2 Rational Bubbles

This model does not allow for one phenomenon that has been observed in asset markets, bubbles

- Tulip mania
- South sea bubble
- Japanese stocks
- High tech stocks

The convergence assumption in (3) is essential for a unique solution to (2). But if we drop that assumption, there are infinite solutions. Any solution can be written as:

\[
P_t = P_{Dt} + B_t
\]

\[
B_t = E_t \left[ \frac{B_{t+1}}{1+R} \right]
\]

Jon Danielsson ©2000
the term is $P_{Dr}$ is known as the **fundamental value** and $B$ as **rational bubble**. e.g.

$$B_{t+1} = \begin{cases} \frac{1 + R}{\pi} B_t + \zeta_{t+1} & \text{with probability } \pi \\ \zeta_{t+1} & \text{with probability } 1 - \pi \end{cases}$$

$$E[\zeta_{t+1}] = 0$$

so a constant probability $1 - \pi$ of bursting each period. Otherwise it grows at rate $\frac{1 + R}{\pi} - 1$ faster than $R$.

### 3 Prices and Returns

Prices of financial assets are not really interesting, since what we care about is the amount of the investment and the return on that investment, not the price of the asset. It makes no difference whether we own 1000 shares in IBM at $100 per share, or 2000 shares at $50. As the price of an asset increases over time each increment in the price is ever larger. While this gives financial journalists the chance to proclaim that a new record was reached on Wall Street, invariably because the DJIA index increased by a record amount, they neglect to tell us the percentage change, often quite puny when compared with other historical records.

![Figure 1: SP-500 Index 1976-1998](image)

Consider the prices\(^1\) of the SP-500 Index in Figure 1. If one judges the market solely from the figure it would seem that the market was very quiet in the mid seventies, compared to now. So is volatility increasing? No. As a result, the primary measurement for financial data is returns.

\(^1\)All data, unless otherwise indicated was obtained from DATASTREAM.

---

Jon Danielsson ©2000
3.1 Returns

Definition 2 (Returns) The relative change in the price of a financial asset over a given time interval, often expressed in percentages is called the return on the asset.

Prices are $P_t$ where $t$ can be any frequency (e.g. daily, yearly, hourly). Now, it would seam that calculating returns would be a trivial affair, but alas, that is not true. There are two ways to measure returns, simple and compound returns. For simplicity, we assume there are no dividends, although they can easily be added to the calculation.

Definition 3 (Simple Return ($R_t$)) Percentage change in prices

$$R_t = \frac{P_t}{P_{t-1}} - 1$$

One often needs to scale returns across time. A multi-period ($k$) return is given by

$$R_t (k) = (1 + R_t) (1 + R_{t-1}) (1 + R_{t-2}) \ldots (1 + R_{t-k+1})$$

$$= \frac{P_t}{P_{t-1}} \cdot \frac{P_{t-1}}{P_{t-2}} \ldots \frac{P_{t-k+1}}{P_{t-k}} - 1 = \frac{P_t}{P_{t-k}} - 1$$

If we use years as the relevant time horizon then if we observe a multi-year return and wish to convert them into annualized return we need to

$$\text{Annualized } [R_t (k)] = \left[ \prod_{j=0}^{k-1} (1 + R_{t-j}) \right]^{1/k} - 1$$

this is in general difficult to work with.

A convenient property of simple returns is that in a portfolio setting, then the following holds

$$R_P = \frac{P_t}{P_{t-1}} - 1$$

$$= \sum_{i=1}^{n} w_i R_i$$

For this reason, when working with portfolios, one uses simple returns. An alternative definition is continuously compounded returns:

Definition 4 (Continuously Compounded Returns ($r_t$)) The logarithm of the gross return

$$r_t = \ln (1 + R_t) = \ln \frac{P_t}{P_{t-1}} = p_t - p_{t-1}$$

Jon Danielsson ©2000
The advantages of the compounded returns (or just returns) show up then moving to multi-period returns

\[ r_t (k) = \ln \left( 1 + R_t (k) \right) = \ln \left( (1 + R_t) (1 + R_{t-1}) (1 + R_{t-2}) \ldots (1 + R_{t-k+1}) \right) \]

\[ = \ln (1 + R_t) + \ln (1 + R_{t-1}) + \ldots + \ln (1 + R_{t-k+1}) \]

\[ = r_t + r_{t-1} + \ldots + r_{t-k+1} \]

However since log of a sum does not equal the sum of logs we have the following problem

\[ r_{P,t} = \ln \left( \frac{P_t}{P_{t-1}} \right) \]

\[ \neq \sum w_i \ln \left( \frac{P_{i,t}}{P_{i,t-1}} \right) \]

This may not be very important, for small returns, e.g. daily

\[ r_p \approx \sum_{i=1}^{N} w_i R_i \]

and as

\[ \lim_{t \to 0} r_p = R_p \]

What difference does it make which is used? Say \( P_t = 1000 \) and \( P_{t-1} = 950 \) then

\[ R_t = \frac{1000}{950} - 1 = 0.0526 \]

\[ r_t = \ln \left( \frac{1000}{950} \right) = 0.0513 \]

Another form of returns that is sometimes used is excess returns.

**Definition 5 (Excess Returns \((Z_t \text{ or } z_t)\).)** Returns in excess of some reference rate, often the risk-free rate, \( r_f \).

\[ Z_t = R_t - R_f \]

\[ z = r_t - r_f \]

We see in Figure 2 the returns from Figure 1

### 3.2 Statistical Description of Returns

#### 3.2.1 Distribution of returns

When people think of a distribution of a random event, they think the normal. Why is that? There are many reasons, perhaps the best is that
1. The assumption is often (approximately) right, perhaps with the right conversion, e.g. log.

2. It makes all mathematics a lot simpler

While the first reason is important, I think the second reason is really the reason for the popularity of the normal. There are several reasons, e.g. the normal is relatively simple, and easy to manipulate mathematically. It also has a natural and simple multivariate representation, which is not the case for most other distributions. Also, it scales naturally, for example in calculations of Value-at-Risk, while non-normal distributions do not. And finally, it is completely described by the first and second moments:

\[
m_1 = E [x] \\
m_2 = E [x^2]
\]

and hence, if we know the mean and variance, (and covariance) there is nothing more to know about the returns. If a random variable (RV) is not normal higher order moments are needed as well, i.e. \( m_i, i > 2 \). Because the moments, themselves are not scale free, and hence comparison is difficult, we normally work with scaled moments, where the mean is removed. The 4 first moments are

<table>
<thead>
<tr>
<th>Name</th>
<th>Equation</th>
<th>Expected value for standard normal</th>
<th>What is it</th>
</tr>
</thead>
<tbody>
<tr>
<td>Mean</td>
<td>( E [x] )</td>
<td>0</td>
<td></td>
</tr>
<tr>
<td>Variance</td>
<td>( E [(x - E [x])^2] = \sigma^2 )</td>
<td>1</td>
<td>The asymmetry in the distribution</td>
</tr>
<tr>
<td>Skewness</td>
<td>( \frac{E [(x - E [x])^3]}{\sigma^3} )</td>
<td>0</td>
<td></td>
</tr>
<tr>
<td>Kurtosis</td>
<td>( \frac{E [(x - E [x])^4]}{\sigma^4} )</td>
<td>3</td>
<td>The tail thickness</td>
</tr>
</tbody>
</table>

Normally, people talk about excess kutrosis, i.e. \( K - 3 \).

Jon Danielsson ©2000
4 Anomalies

People have found that markets have some peculiar properties. For example, the following are some of the results obtained.

- Monday effect, returns on Mondays are lower than for other days. A study with 1962-1978 daily US data, Monday return was -33% annualized. Of that, more than half is over the weekend, and the rest in the first 45 minutes of the opening.

- January returns. Highest returns in the year, especially for small stocks, both US and International.
  - One possible explanation, the last trade of the year was at the bid which causes return to appear higher in the first days of the year. e.g. say \( P_{bid} = 20 \), \( P_{ask} = 20\frac{1}{4} \), then the last trade of the year likely to be at 20, and the first trade in January between 20 and 20\(\frac{1}{4} \), say on average 20\(\frac{1}{8} \). Therefore, without any change in bid or ask, return of \( \frac{1}{8}/20 \) per day, which gives a very large annualized return.
  - Another is taxes. Lets say we sell a losing stock in December, realize the loss, buy back in January. However this is difficult to reconcile. The January effect also existed before income taxes, and also in Australia where the tax year starts in April.

- Government bonds: If we look at the yield curve of government T-bills maturing at time \( T \), i.e. \( y(T) \) or, equivalently on the prices per dollar of face value, \( B(T,t) \), then using standard NPV

\[
B(T,t) = y(T)^{T-t}
\]

If we now draw calendar time along the \( x \)-axis (\( t \) is today) and yields along the \( y \)-axis then empirically we observe the following:

\[
\begin{array}{c}
\vdots \\
y_{Dec1} \\
\vdots \\
y_{Jan1} \\
\vdots \\
T
\end{array}
\]

The yield goes up for bills maturing after Jan. 1 (or the price goes down for bills maturing after Jan. 1)! This is likely a TAX effect because bonds maturing after Jan. 1, delay the tax payment on the interest by one year. This should NOT be due to risk premium that changes in the time to maturity because the bills are government bonds which are essentially risk free (or we have no reason to suspect that the US government’s bankruptcy probability varies with time).

Indeed, there is a large number of such perceived anomalies. This would suggest that one could exploit the patterns in trading, as in technical trading. Many investors search for these patterns, and there are many firms and individuals that for a fee are willing to help other people getting rich from such patterns. Sullivan et. al. (1998) lists several that have been reported in the (pseudo) academic literature: (the discussion is also based on Sullivan)

Jon Danielsson ©2000
• day of the week
• week of the month
• month of the year
• turn of the month
• turn of the year
• holiday effects
• etc., etc., etc.

These are found by researchers that systematically mine available data to find patterns. Are these patterns real or an illusion. Sullivan analyze whether the patterns are real or as a result of data snooping. The problem is similar to data mining problems, by choosing the best out of a group, and conditioning some analysis on that, the significance level is different that if one had not searched. They consider a large set of calendar rules, and estimate the real significance level. The result is that while a researcher might find a significant calendar effect, in actuality, all calendar effects were non-significant.

5 Predictability of Returns (CLM ch. 2)

5.1 Random Walk Hypothesis (RWH)

The best definition of random walk (RW) is the path a sufficiently inebriated person follows. In the simplest form it says that

\[ P_t = P_t + \varepsilon_t \]
\[ E [P_{t+1}] = P_t \]

i.e. that prices fluctuate around a fixed mean. We see more of this below, where we allow for more complicated versions, but leave that aside for a moment. But does this hold?

\textbf{Efficient markets }\iff\textbf{ Prices follow a form of random walk}

There are many obvious reasons why the RWH is neither a necessary nor sufficient condition for the rational determination of prices. Under risk neutrality they may, but that is silly. The reason is e.g. that risk matters. If we are faced with one asset where future dividends are knows, think government bonds in some countries, versus risky corporate dividends, if the risky and riskless dividends have the same expected value, which do you prefer?

Indeed, Grossman and Stiglitz argue that if markets were perfectly efficient, markets would collapse, since there were no payoff in obtaining information, and trading would cease. In a weaker form, the degree of market efficiency reflects the cost of obtaining information, and the resulting profits are rent. Provided perhaps from noise trades as suggested by Black...

Jon Danielsson ©2000
(1986). If a person is trading for liquidity reasons, they may be considered equivalent to noise traders, but are they irrational?

CLM discuss various models of random walk (RW) and how one can test for random walk, i.e. predictability of returns. We will only discuss some of the simplest here. However, if you are interested, the discussion in CLM is detailed.

### 5.2 Random walk

Define two functions \( f(r_t) \) and \( g(r_{t+1}) \) and consider

\[
\text{Cov} [f(r_t), g(r_{t+k})] = 0 \text{ all } t \text{ and } k \neq 0
\]

for an appropriate choice of functions, we can capture almost all random walk, and martingale hypothesis with this orthogonality condition, see below

#### 5.2.1 Martingale Model

This is the oldest model of prices, i.e. that they are like a fair game

\[
E[P_{t+1}|P_t, P_{t-1}, ..] = P_t
\]

\[
E[P_{t+1} - P_t|P_t, P_{t-1}, ..] = 0
\]

So, today’s price best in forecasting. Therefore, non-overlapping changes are uncorrelated at all leads and lags, and therefore implies linear forecasting rules are infective. This used to be considered a necessary condition of efficient markets, but by considering the risk-return trade-off, it has been shown that the martingale property is neither a necessary nor a sufficient condition for rationally determined asset prices. But once when adjusted for risk, it holds (equivalent martingale measures)

#### 5.2.2 Random Walk

There are several formulations of a model how prices change, each more general than others:

**Random Walk I**  IID increments

\[
P_t = \mu + P_{t-1} + \epsilon_t \quad \epsilon_t \sim \text{IID} \left(0, \sigma^2\right)
\]

where \( \mu \) is drift. The following follows trivially

\[
E[P_t|P_0] = P_0 + \mu t
\]

\[
\text{Var} [P_t|P_0] = \sigma^2 t
\]

This also holds for the other two RW models. What distributions to use? The normal as in arithmetic Brownian motion, but there are problems with limited liability, i.e. \( \text{Pr}[P < 0] > 0 \)

Jon Danielsson ©2000
Most common, log-normality.

\[ p_t = \mu + p_{t-1} + \varepsilon_t \quad \varepsilon_t \sim \mathcal{N} \left( 0, \sigma^2 \right) \]

i.e. log normal diffusion.

5.2.3 Random Walk II

The basic problem with RWI is first that the assumption iid increments not believable. For example.

for the past two hundred years since markets started, the probability law must have changed (reasons many). So we relax RWI to allow for INID (independent not identically)

5.2.4 Random Walk III

Allow for dependent but uncorrelated increments. An example of a process which satisfies RW3 and nor RW1 or RW2 is where

\[ \text{Cov} \left[ \varepsilon_t, \varepsilon_{t-k} \right] = 0 \quad \text{all } k \neq 0 \]
\[ \text{Cov} \left[ \varepsilon_t^2, \varepsilon_{t-k}^2 \right] \neq 0 \quad \text{for some } k \neq 0 \]

i.e. uncorrelated in first moment, but not the second, or squared increments. What this says is that the volatility (variance) of returns may be dependent, but not the mean, i.e. we can forecast volatility and not prices. This is in line with current hypothesis about the markets.

5.3 Tests of RWIII

I want to discuss some of the simpler tests of RWIII.

5.3.1 Autocorrelation Coefficients

\[ \text{Corr} \left[ x, y \right] = \frac{\text{Cov} \left[ x, y \right]}{\sqrt{\text{Var} \left[ x \right]} \sqrt{\text{Var} \left[ y \right]}} \]

Jon Danielsson ©2000
and if series is covariance stationary, i.e. has second moment autocovariance and autocorrelation respectively:

\[
\gamma (k) \equiv \text{Cov} [r_t, r_{t+k}]
\]

\[
\rho (k) = \frac{\text{Cov} [r_t, r_{t+k}]}{\sqrt{\text{Var} [r_t]} \sqrt{\text{Var} [r_{t+k}]}} = \frac{\gamma (k)}{\gamma (0)}
\]

these can be estimated by population moments

\[
\hat{\gamma} (k) = \frac{1}{T} \sum_{t=1}^{T} (r_t - \bar{r}_T) (r_{t+k} - \bar{r}_T) \quad 0 \leq k < T
\]

\[
\hat{\rho} (k) = \frac{\hat{\gamma} (k)}{\hat{\gamma} (0)}
\]

\[
\bar{r}_T = \frac{1}{T} \sum_{t=1}^{T} r_t
\]

5.3.2 Portmanteau Statistic (Q)

A simple test of RWI (which implies all autocorrelations are zero) is the Box-Pierce

\[
Q_m = T \sum_{k=1}^{m} \rho^2 (k)
\]

It is easy to show under RWI that \( \hat{Q} \) is \( \chi^2 \) a better version for small samples is Ljung–Box

\[
Q'_m \equiv T (T + 2) \sum_{k=1}^{m} \frac{\rho^2 (k)}{T - k} \sim \chi^2_m
\]

simply summing up of correlation. One problem is selecting the \( m \)

5.3.3 Unit Root Tests

Another way is unit root tests for stationarity.

5.3.4 Stationarity.

If process is in equilibrium then it is stationary. If it is strictly stationary then its statistical properties are identical whenever you look at the data, i.e. the distribution function of \( (x_1, x_2, \ldots, x_m) \) is identical to the data at time \( t+k \). Also implies that if \( E |x_t|^2 < \infty \) the mean and variance are fixed

\[
E (x_1) = E (x_2) = E (x_T) = \mu
\]

\[
V (x_1) = V (x_2) = V (x_T) = \sigma^2_x
\]

\[
\text{Cov} (x_1, x_{1+k}) = \text{Cov} (x_2, x_{2+k}) = \ldots = \text{Cov} (x_{T-k}, x_T)
\]

Jon Danielsson ©2000
also that the autocovariance is fixed
\[ \gamma_k = \text{Cov}(x_{t-k}, x_t) = E[(x_t - \mu)(x_{t-k} - \mu)] \]
and autocorrelation:
\[ \rho_k = \frac{\text{Cov}(x_{t-k}, x_t)}{\sqrt{\text{Var}(x_t) \text{Var}(x_{t-1})}} = \frac{\gamma_k}{\gamma_0} \]
is only dependent on \( k \). This is also known as week stationarity. Strict stationarity \( \Rightarrow \) week stationarity., not vice versa. If normal they are the same